THE GROMOV WIDTH OF SYMPLECTIC CUTS OF SYMPLECTIC MANIFOLDS

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In 1985, Gromov discovered a rigidity phenonmenon for symplectic embeddings which led to the concept of Gromov width: a measure of the largest ball that can be symplectically embedded inside a symplectic manifold. This is an invariant of the symplectic form. The central theme of this thesis is computing the Gromov width of symplectic cuts. We do this for two classes of symplectic manifolds: four-dimensional toric manifolds and complex Grassmannian manifolds. Symplectic cutting results in symplectic manifolds of smaller volume. A natural question is whether the Gromov width decreases (or at least does not increase) under this operation. In this thesis we use symplectic embedding techniques and theory of J-holomorphic curves to establish lower and upper bounds on Gromov width. In the case of 4-dimensional toric symplectic manifolds, we answer the question positively for any symplectic cuts that result in smooth manifolds. We then compute the exact Gromov width of certain cuts of complex Grassmannians, again establishing the desired monotonicity.

BIOGRAPHICAL SKETCH

Huynh was born on February 7 in 1990 in Tan An - Long An. Back then it was a small town 30 miles southwest of Ho Chi Minh city in Vietnam. My grew up in such a poor family that they could barely afford daily meals or his school tuitions. In 2005, they moved to the United States with his parents's main hope that he would be able to study and take full advantage of educational opportunities that the United States has to offer.

His family settled in Arizona and he attended La Joya Community High School in Avondale, Arizona from 2005 to 2008. From 2008 to 2012, My studied mathematics at Arizona State University and he completed the Mathematics Advanced Study Semesters (MASS) program at Penn State University in fall 2009. As a undergraduate, he spent all his summers participating in NSF REU programs at Arizona State University and Iowa Sate University. These events inspired him to obtain a Ph.D. degree in mathematics. In 2012, My started his Ph.D. program at Cornell University. Throughout his educational career, he has met many wonderful mentors and without them, he would not be at this stage of his career.

To my parents for always encouraging me,

and

to my wife for always supporting me.

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CHAPTER 1

INTRODUCTION

The Darboux theorem in symplectic geometry states that for any point in a symplectic manifold, there is a system of local coordinates such that the symplectic manifold looks locally like $\mathbb{R}^{2n}$ equipped with its standard symplectic form. Over the past thirty years, researchers have been interested in a natural and fundamental problem in symplectic geometry: how far we can symplectically extend these coordinates in the symplectic manifold? In other words, what is the largest size of a ball embedded symplectically into the symplectic manifold? This is how the concept of Gromov width arises.

Let $B^{2n}(r) = \left\{(x_1, \ldots, x_n, y_1, \ldots, y_n) \,\middle|\, \sum_{i=1}^{n}(x_i^2 + y_i^2) < r\right\}$ denote the open ball of radius r centered at the origin in $\mathbb{R}^{2n}$ and $\omega_{std} := \sum_{i=1}^{n} dx_i \wedge dy_i$ the standard symplectic form on $\mathbb{R}^{2n}$, which we may restrict to $B^{2n}(r)$. A symplectic embedding ϕ of $(B^{2n}(r), \omega_{std})$ into a symplectic manifold (M^{2n}, ω), denoted by $\phi : B^{2n}(r) \overset{s}{\hookrightarrow} M^{2n}$, is a smooth embedding ϕ of $B^{2n}(r)$ into M^{2n} satisfying $\phi^*\omega = \omega_{std}$. The **Gromov width** of a symplectic manifold (M, ω) is defined as

$$\mathrm{Gwidth}(M) := \sup\{\pi r^2 \mid B^{2n}(r) \overset{s}{\hookrightarrow} M^{2n}\}.$$

The Darboux theorem guarantees that the Gromov width of any symplectic manifold is always positive.

The first result of this type is Gromov's celebrated Non-squeezing Theorem. He developed J-holomorphic techniques to provide upper bounds on the invariant now called Gromov width.

Theorem 1.0.1 (Gromov's non-squeezing theorem, [12]). *If there is a symplectic* *embedding of the ball $B^{2n}(r)$ of radius r into a cyclinder $B^2(\lambda) \times \mathbb{R}^{2n-2}$ of radius λ, then $r \leq \lambda$.*

In particular,

$$\mathrm{Gwidth}(B^2(\lambda) \times \mathbb{R}^{2n-2}) = \pi\lambda^2.$$

The Non-squeezing Theorem was proved by Gromov in 1985, where a beautiful application of the theory of J-holomorphic curves to symplectic geometry is presented. Since then, many authors have used Gromov's method for bounding from above the Gromov width of many families of symplectic manifolds including in the works of Lu for toric symplectic manifolds [27], Lu and Karshon-Tolman for complex Grassmannian manifolds [26] and [22], respectively, Mandini and Pabiniak for polygon spaces [29], Zoghi for regular coadjoint orbits [41], and Caviedes Castro extended Zoghi's results to all coadjoint orbits of compact Lie Groups [7].

Lerman introduced symplectic cutting, a technique used to split a symplectic manifold into two new symplectic manifolds, each with smaller volume than the first. It is natural to ask whether or not the Gromov width of a symplectic cut is at most the Gromov width of the original manifold. Computing the Gromov width of a symplectic cut is itself a fascinating problem. In this thesis we will deploy various techniques to compute the Gromov width of a symplectic cut and compare the result with the Gromov width of the original manifold. Under symplectic cutting, the volume of a symplectic cut is less than the volume of the original manifold. One would think that the Gromov width should behave similarly; however, due to rigidity of symplectic embeddings, the Gromov width needs not decrease when the volume decreases.

We will use symplectic cutting to produce new compact symplectic manifolds from old compact ones equipped with sufficient symmetries. If (M, ω) is a connected symplectic manifold, then the action of a (compact) torus $\mathbb{T}^n \cong (S^1)^k$ is called **Hamiltonian** if there exists a $\mathbb{T}$-invariant map $\Phi : M \to \mathfrak{t}^*$, called the **moment map**, such that

$$i(\xi_M)\omega = -d\langle \Phi, \xi \rangle \quad \forall \xi \in \mathfrak{t},$$

where ξ_M is the vector field on M generated by $\xi \in \mathfrak{t}$. For the rest of the thesis, we identify $\mathfrak{t}$ and $\mathfrak{t}^*$ with $\mathbb{R}^n$.

Example 1.0.2. On the manifold $(\mathbb{C}, \frac{i}{2} dz \wedge d\bar{z})$, consider the action of the circle $S^1 = \left\{ t \in \mathbb{C} \,\middle|\, |t| = 1 \right\}$ by rotations

$$\psi_t(z) = t^k z \ \text{ for } t \in S^1,$$

where $k \in \mathbb{Z}$ is fixed. The action $\phi : S^1 \to \mathrm{Diff}(\mathbb{C})$ is Hamiltonian with moment map $\mu : \mathbb{C} \to \mathbb{R}$ given by

$$\mu(z) = \frac{1}{2} k |z|^2.$$

In polar coordinates, $\omega_0 = r dr \wedge d\theta$ so $\mu(re^{i\theta}) = \frac{1}{2} k r^2$ and the vector field on $\mathbb{C}$ corresponding to the generator $\mathbf{1}$ of $\mathfrak{g} \simeq \mathbb{R}$ is $\xi_M = k \frac{\partial}{\partial \theta}$. Then

$$i(\xi_M)\omega_0 = -krdr = -d\left(\frac{1}{2} k r^2 \right).$$

Let (M, ω) be a connected symplectic manifold with a Hamiltonian circle action (for $\mathbb{T} = S^1$) and a moment map $\mu : M \to \mathbb{R}$. Suppose that ϵ is a regular value of the moment map and the circle acts freely on the level set $\mu^{-1}(\epsilon)$. We consider the disjoint union

$$\widetilde{M} = M_{[\epsilon, \infty)} := \mu^{-1}((\epsilon, \infty)) \sqcup M_\epsilon,$$

obtained from the manifold with boundary $\mu^{-1}([\epsilon,\infty))$ by collapsing the boundary under the S^1-action where $M_\epsilon = \mu^{-1}(\epsilon)/S^1$. Lerman [25] shows that $\widetilde{M}$ is naturally a symplectic manifold, in such a way that the embeddings of $\mu^{-1}((\epsilon,\infty))$ and M_ϵ are symplectic. Moreover, the induced circle action on $\widetilde{M}$ is Hamiltonian, with the moment map coming from the restriction of the original moment map μ to $\{m \in M : \mu(m) \geq \epsilon\}$. The symplectic manifold $\widetilde{M}$ is called the **symplectic cut** of M with respect to the ray $[\epsilon,\infty)$. The operation that produces the symplectic cut $\widetilde{M}$ is called **symplectic cutting**. We will consider the symplectic cut with respect to the ray $[\epsilon,\infty)$ unless otherwise stated, simply call the symplectic cut of M. The symplectic cut with respect to the ray $(-\infty,\epsilon]$ is equally interesting, and results in this thesis are applied to this cut as well. At a regular value ϵ, the circle action has at worst finite stabilizers. When the action is not free, $\widetilde{M}$ is a symplectic **orbifold**.

Example 1.0.3. Consider the symplectic 2-sphere $(M = S^2, d\theta \wedge dh)$ in cylindrical coordinates, the parameter group of diffeomorphisms given by rotation around the vertical axis, $\psi_t(\theta,h) = (\theta + t, h)(t \in \mathbb{R})$ is a Hamiltonian action of the group $S^1 \simeq \mathbb{R}/\langle 2\pi \rangle$, as it preserves the area form $d\theta \wedge dh$. The moment map $\mu : S^2 \to \mathbb{R}$ is described in Figure 1.1. We apply symplectic cutting on this manifold at the regular value ϵ. M_ϵ is the upper hemisphere of the 2-sphere, the rim of the hemisphere $\mu^{-1}(\epsilon)$ is the circle equator of the 2-sphere. Note that S^1 acts freely on $\mu^{-1}(\epsilon)$ so $\mu^{-1}(\epsilon)/S^1$ collapses to a point. The new symplectic cut $\widetilde{M} = M_{[\epsilon,\infty)}$ is another 2-sphere whose south pole is at the level of ϵ, which is the red sphere in Figure 1.1.

In this thesis we are interested in finding the Gromov width of symplectic cuts of two classes of symplectic manifolds: four-dimensional toric symplectic manifolds and complex Grassmannian manifolds. The rest of the chapters are

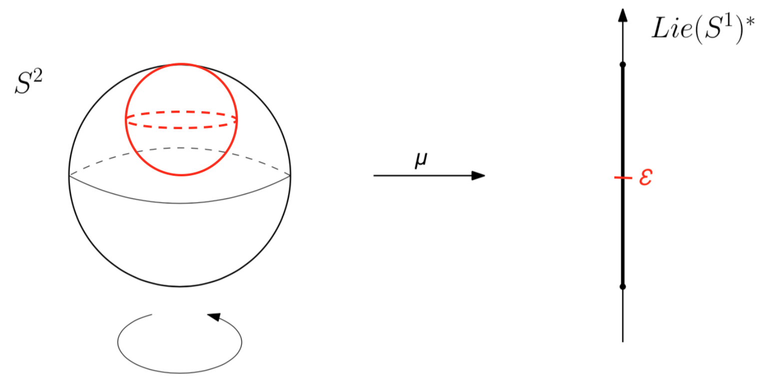


Figure 1.1: Hamiltonian S^1 action on two-sphere S^2 and its symplectic cut in red color

divided as follows: Chapter 2 provides background materials and surveys on relevant previous work, Chapter 3 computes the Gromov width of all smooth symplectic cuts of four-dimensional toric symplectic manifolds and Chapter 4 computes the Gromov width of certain symplectic cuts of complex Grassmannian manifolds.

CHAPTER 2

BACKGROUND

In this chapter, we develop the background needed for this thesis and we also survey relevant previous work in symplectic geometry. This includes background materials on symplectic cutting, toric symplectic manifolds, equivariant techniques, and theory of J-holomorphic curves.

2.1 Properties of symplectic cutting

In this section, we quote results from [25] that the symplectic cuts at regular values of a free Hamiltonian circle action are symplectic manifolds. Basic properties of symplectic cutting are also surveyed.

Suppose that (M, ω) is a symplectic manifold with a Hamiltonian circle action and a moment map $\mu : M \to \mathbb{R}$ and further suppose that the circle S^1 acts freely on the level set $\mu^{-1}(\epsilon)$. The product manifold $(M \times \mathbb{C}, \omega \oplus \frac{1}{i} dw \times d\bar{w})$ is equipped with a Hamiltonian circle action S^1 defined by

$$e^{i\theta}(m, w) = (e^{i\theta} m, e^{-i\theta} w)$$

with a moment map

$$\Phi(m, w) = \mu(m) - |w|^2.$$

Then ϵ is a regular value of Φ and S^1 also acts freely on this level set. By the Marsden-Weinstein-Mayer Theorem [6, Theorem 23.1] on symplectic quotients, $\Phi^{-1}(\epsilon)/S^1$ is again a symplectic manifold. The set $\{\Phi = \epsilon\}$ is a disjoint union of two S^1 invariant manifolds:

$$\{\Phi = \epsilon\} = \left\{ (m, w) \,\middle|\, \mu(m) > \epsilon \;\&\; w = e^{i\theta}\sqrt{\mu(m) - \epsilon} \right\} \bigsqcup \left\{ (m, 0) \,\middle|\, \mu(m) = \epsilon \right\}.$$

The first manifold is equivariantly diffeomorphic to the product of $M_{(\epsilon,\infty)} = \{m : \mu(m) > \epsilon\}$ and of the circle S^1 and the second manifold is diffeomorphic to the ϵ level set $\mu^{-1}(\epsilon)$. Consequently, the manifold $M_{(\epsilon,\infty)}$ embeds as an open dense submanifold into the reduced space

$$M_{[\epsilon,\infty)} := \Phi^{-1}(\epsilon)/S^1 = \left\{(m,w) \in M \times \mathbb{C} \,\middle|\, \mu(m) - |w|^2 = \epsilon\right\}/S^1$$

and the difference $M_{[\epsilon,\infty)} - M_{(\epsilon,\infty)}$ is symplectomorphic to the reduced space $\mu^{-1}(\epsilon)/S^1$. Thus, the symplectic cut $M_{[\epsilon,\infty)}$ is a symplectic manifold equipped with a Hamiltonian circle action S^1 inherited from M. We end this section with several main properties of symplectic cutting in the next few remarks.

Remark 2.1.1 (Cuts and Hamiltonian group actions)**.** If in addition to the action of a circle on our manifold (M, ω) we have a Hamiltonian action of another group K on M that commutes with the action of S^1 then the symplectic cut is again a Hamiltonian K-space.

Remark 2.1.2 (Cuts and global blow-ups)**.** Let (M, ω) be a Hamiltonian S^1 space with a proper momentum map $\mu : M \to \mathbb{R}$. Suppose the moment map achieves its maximum on M and that it achieves this maximal value c at a single point m_0. Then for ϵ sufficiently small, m_0 is the only critical point in the set $M_{\mu > c-\epsilon} = \{m \in M \mid c - \mu(m) < \epsilon\}$. Assume further that the weights of the isotropy representation of S^1 on $T_{m_0}M$ are all 1. It follows from the equivariant Darboux theorem that the level sets $\mu = c - \delta$ are spheres for all $0 < \delta < \epsilon$. Therefore, the manifold $M_{\mu > c-\epsilon}$ is the **blow-up** of M at m_0 by a δ amount since we obtained it by removing a set symplectomorphic to an open ball and then collapsing the fibers of the Hopf fibration in the boundary of the remaining set.

Remark 2.1.3 (Cuts and moment polytopes)**.** Atiyah-Guillemin-Sternberg convexity theorem [1], [14] says that if a torus T acts on a compact symplectic manifold (M, ω) with a moment map $\mu : M \to \mathfrak{t}^*$, then $\mu(M)$ is a rational convex polytope in $\mathfrak{t}^*$. Suppose $\xi \in \mathfrak{t}$ generates a circle subgroup S_ξ of T. Then the action of S_ξ on (M, ω) is Hamiltonian with moment map $\mu^\xi = \xi \circ \mu$. The actions of S_ξ and of T commute. If we cut M at $\epsilon \in \mathbb{R}$ using μ^ξ ,we get a Hamiltonian T orbifold. Its moment polytope is

$$\mu(M) \cap \{\lambda \in \mathfrak{t}^* \mid \langle \xi, \lambda \rangle \geq \epsilon\}.$$

Thus, there is a correspondence between symplectic cuts of manifolds and cuts of moment polytopes.

2.2 Toric symplectic manifolds

A $2n$-dimensional toric symplectic manifold (M^{2n}, ω) is a compact, connected symplectic manifold equipped with an effective Hamiltonian action of an n-dimensional torus (the dimension of the torus is half the dimension of the manifold) with a corresponding moment map $\mu : M \to \mathbb{R}^n$. The coadjoint action is trivial on a torus. Hence, if $\mathbb{T}^n$ is an n-dimensional torus with Lie algebra and its dual both identified with Euclidean space, $\mathfrak{g} \simeq \mathbb{R}^n$ and $\mathfrak{g}^* \simeq \mathbb{R}^n$, a moment map for an action of $\mathbb{T}^n$ on (M, ω) is simply a map $\mu = (\mu_1, \ldots, \mu_n) : M \to \mathbb{R}^n$ satisfying:

- For each basis vector X_i of $\mathbb{R}^n$, the function μ_i is a Hamiltonian function for the vector field ξ_i on M generated by X_i satisfying $i_{\xi_i}\omega = -d(\mu_i)$ and is invariant under the action of the torus.

The Atiyah/Guillmeimin-Sternberg convexity Theorem guarantees that $\mu(M)$ is a convex polytope.

Theorem 2.2.1 (Atiyah [1], Guillemin-Sternberg [14]). *Let (M,ω) be a compact connected symplectic manifold, and let $\mathbb{T}^m$ be an m-torus. Suppose that $\psi : \mathbb{T}^m \to \mathrm{Sympl}(M,\omega)$ is a Hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$ where $\mathrm{Sympl}(M,\omega)$ is the space of all symplectomorphisms from M to itself. Then*

a) the levels of μ are connected;

b) the image of μ is convex;

c) the image of μ is the convex hull of the images of the fixed points of the action.

Thus the image $\mu(M)$ of the moment map is called the **moment polytope** and it is convenient to think that the moment polytope lies in $\mathbb{R}^n$ via the identification above.

The Example 1.1 is an example of a symplectic manifold whose moment image is a line segment in $\mathbb{R}$. Below is a more interesting example whose moment polytope is a triangle in $\mathbb{R}^2$.

Example 2.2.2. Let $(\mathbb{CP}^2, \omega_{FS})$ be the 2-dimensional complex projective plane equipped with the Fubini-Study form [5, Section 2.3]. The $\mathbb{T}^2$-action on $\mathbb{CP}^2$ by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$ has moment map

$$\mu[z_0 : z_1 : z_2] = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right).$$

The fixed points have images

$$\begin{aligned}
\mu([1:0:0]) &= (0,0),\\
\mu([0:1:0]) &= (\tfrac{1}{2},0),\\
\mu([0:0:1]) &= (0,\tfrac{1}{2}).
\end{aligned}$$

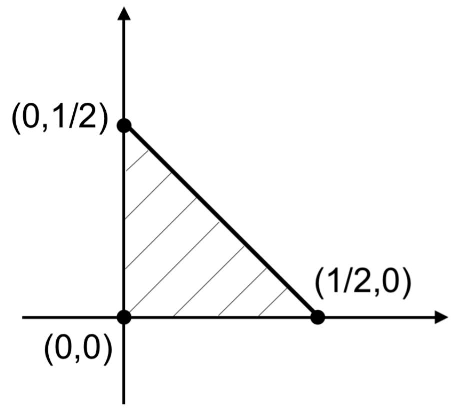


Figure 2.1: The moment polytope for the standard $\mathbb{T}^2$ action on $\mathbb{CP}^2$

Delzant's theorem [9] completely classifies toric symplectic manifolds up to equivariant symplectomorphism by the combinatorial data encoded in the corresponding Delzant polytope. We first define a Delzant polytope.

Definition 2.2.3. A **Delzant polytope** Δ in $\mathbb{R}^n$ is a polytope satisfying:

- **simplicity**: there are n edges meeting at each vertex;
- **rationality**: the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i, t \geq 0$, where $u_i \in \mathbb{Z}^n$;
- **smoothness**: for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Example 2.2.4. The figures below show examples of Delzant polytopes and polytopes that are not Delzant in $\mathbb{R}^2$.

Theorem 2.2.5 (Delzant [9])**.** Toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets

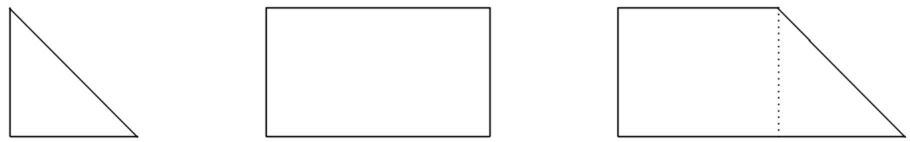


Figure 2.2: Examples of Delzant polytopes in $\mathbb{R}^2$.

Figure 2.3: Examples of polytopes that are not Delzant: the polytope on the left fails the smoothness condition while the other fails the simplicity condition.

of equivalence classes is given by the moment map:

$$\begin{array}{ccc} \dfrac{\{\text{toric manifolds}\}}{\{T^n \text{ equivariant symplectomorphisms}\}} & \xrightarrow{1-1} & \dfrac{\{\text{Delzant polytopes}\}}{\{\text{translations}\}} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) & \longrightarrow & \mu(M). \end{array}$$

2.3 Symplectic embeddings of a ball using equivariant techniques

In this section we review the equivariant technique that Karshon and Tolman developed to construct symplectic embeddings of open subsets of $\mathbb{C}^n$ into symplectic manifolds with a Hamiltonian torus action; for a detailed treatment of these results, see [22]. Let a torus $\mathbb{T} \cong (S^1)^{\dim \mathbb{T}}$ with Lie algebra $\mathfrak{t}$ act effectively on a connected symplectic manifold (M, ω). A moment map is a map $\Phi : M \to \mathfrak{t}^*$

Figure 2.4: Isotropy weights of standard S^1 action on $\mathbb{C}^2$ by rotation

such that

$$i(\xi_M)\omega = -d\langle \Phi, \xi \rangle \ \forall \xi \in \mathfrak{t}$$

where ξ_M is the corresponding vector field on M.

Let p be a fixed point. There exists $\eta_j \in \mathfrak{t}^*$, called the **isotropy weights** at p, such that the induced linear symplectic $\mathbb{T}$-action on the tangent space T_pM is isomorphic to the action on $(\mathbb{C}^n, \omega_{std})$ generated by the moment map

$$\Phi_{\mathbb{C}^n}(z) = \Phi(p) + \pi \sum |z_j|^2 \eta_j. \tag{2.1}$$

The isotropy weights are uniquely determined up to permutation. Note that with our sign convention in equation 2.1 the isotropy weights are pointing into the moment image. For example, the standard S^1 action on $\mathbb{C}^2$ by rotation with speed one gives the following moment image in Figure 2.4.

The equivariant Darboux theorem [40] tells us that a neighborhood of p in M is equivariantly symplectomorphic to a neighborhood of 0 in $\mathbb{C}^n$. However this theorem does not tell us how large we can take this neighborhood to be. Karshon and Tolman developed an equivariant technique that allows us to control the size of this neighborhood.

Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex set which contains $\Phi(M)$. The quadruple $(M, \omega, \Phi, \mathcal{T})$ is a proper Hamiltonian T-manifold if Φ is proper as a map to $\mathcal{T}$, that is, the preimage of every compact subset of $\mathcal{T}$ is compact.

Definition 2.3.1. A proper Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ is **centered** about a point $\alpha \in \mathcal{T}$ if α is contained in the moment map image of every component of M^K, for each $K \subset T$.

We now quote several examples and non-examples, see [22] for more details.

Example 2.3.2. A compact symplectic manifold with a non-trivial torus action is never centered because it has fixed points with different moment images.

Example 2.3.3. Let a torus T act linearly on $\mathbb{C}^n$ with a proper moment map $\Phi_{\mathbb{C}^n}$ such that $\Phi_{\mathbb{C}^n}(0) = 0$. Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex subset containing the origin. Then $\Phi_{\mathbb{C}^n}^{-1}(\mathcal{T})$ is centered about the origin.

Proposition 2.3.4. [22, Proposition 2.8] *Let $(M, \omega, \Phi, \mathcal{T})$ be a proper Hamiltonian T-manifold. Assume that M is centered about $\alpha \in \mathcal{T}$ and that $\Phi^{-1}(\{\alpha\})$ consists of a single fixed point p. Then M is equivariantly symplectomorphic to*

$$\{z \in \mathbb{C}^n \mid \alpha + \pi \sum |z_j| \eta_j \in \mathcal{T}\},$$

where $\eta_1, \ldots, \eta_n$ are the isotropy weights at p.

We quote the following example that demonstrates the idea of the Proposition 2.3.4 from [36]

Example 2.3.5. Consider a compact toric symplectic manifold M whose moment map image is the closure of the following region in Figure 2.5.

The isotropy weights of the torus action are η_1 and η_2 and the lattice lengths of edges starting from α are 5 and 2 (with respect to lattice of isotropy weights).

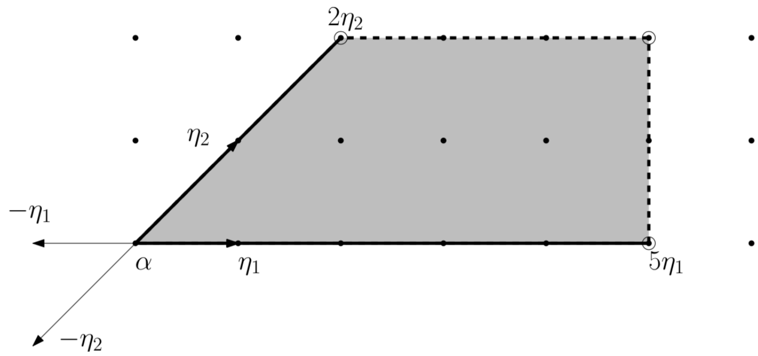


Figure 2.5: Example of an equivariant ball

The largest subset of M that is centered about α maps under the moment map to the shaded region. The Proposition above tells us that it is equivariantly symplectomorphic to

$$\{z \in \mathbb{C}^2 \mid \alpha + \pi(|z_1|^2\eta_1 + |z_2|^2\eta_2) \in \text{shaded region}\}.$$

If $z \in B^4(2) = \{z \in \mathbb{C}^2 \mid \pi(|z_1|^2 + |z_2|^2) < 2\}$ then $\alpha + \pi(|z_1|^2\eta_1 + |z_2|^2\eta_2)$ is in the shaded region. Therefore the 4-dimensional ball $B^4(2)$ of capacity 2 embeds into M and the Gromov width of M is at least the minimum of lattice lengths of edges of the moment polytope, starting at α. Note also that the moment map image of the embedded ball $B^4(2)$ is the triangle with vertices $\alpha, \alpha + 2\eta_1$ and $\alpha + 2\eta_2$.

2.4 Theory of J-holomorphic curves and Gromov-Witten invariants

In this section we will give a brief review of J-holomorphic theory and Gromov-Witten invariants. We show how J-holomorphic curves are used to compute upper bounds for Gromov width of a symplectic manifold. Most of the materials presented here is adapted from [7] and [32] .

Let (M^{2n}, ω) be a $2n$-dimensional symplectic manifold. An almost complex structure J of (M, ω) is a smooth operator $J : TM \to TM$ such that $J^2 = -I$ where I is the identity operator on TM. An almost complex structure J is said to be **integrable** if it arises from an underlying complex structure on M. We say that an almost complex structure J is **compatible** with ω if the formula

$$g(v, w) := w(v, Jw)$$

defines a Riemannian metric. We denote the space of ω-compatible almost complex structures by $\mathcal{J}(M, \omega)$.

Let $(\mathbb{CP}^1, j)$ be the Riemann sphere with its standard complex structure j and $J \in \mathcal{J}(M, \omega)$. A map $u : \mathbb{CP}^1 \to M$ is called a J**-holomorphic curve of genus zero** or simply a J**-holomorphic curve** if

$$J \cdot du = du \cdot j,$$

or equivalently if $\bar{\partial}_J(u) = 0$ where $\bar{\partial}_J$ is the operator defined by

$$\bar{\partial}_J(u) = (du + J \cdot du \cdot j).$$

A J-holomorphic curve $u : \mathbb{CP}^1 \to M$ is said to be **multiply covered** if there is J-holomorphic curve $u' : \mathbb{CP}^1 \to M$ and a holomorphic branched covering

$\phi : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree greater than one such that

$$u = u' \cdot \phi.$$

It is **simple** if it is not multiply covered.

For a second holomogy class $A \in H_2(M;\mathbb{Z})$, we define the **moduli space of simple J-holomorphic curves of degree A** as

$$\mathcal{M}_A(M,J) = \{u : \mathbb{CP}^1 \to M \ : \ J \cdot du = du \cdot j, u_*[\mathbb{CP}^1] = A, u \text{ is simple}\}.$$

For a generic almost complex structure J the moduli space $\mathcal{M}_A(M,J)$ is an oriented smooth manifold of dimensional equal to

$$\dim M + 2c_1(TM)(A),$$

where c_1 denotes the first Chern class of the bundle (TM,J) (see e.g. [32, Theorem 3.1.5]).

Let $\mathcal{M}^*_{A,k}(M,J)$ denote the set of equivalence classes, under the action of the reparametrization group $PSL(2,\mathbb{C})$ of simple J-holomorphic maps

$$u : (\mathbb{CP}^1, z_1, \ldots, z_k) \to M$$

of degree A with k marked distinct points $z_i \in \mathbb{CP}^1$. For a generic J, the moduli space $\mathcal{M}^*_{A,k}(M,J)$ is a smooth oriented manifold of dimension equal to

$$\dim M + 2c_1(TM)(A) + 2k - 6.$$

An **evaluation map**

$$\mathrm{ev}^k_J = (\mathrm{ev}_1, \ldots, \mathrm{ev}_k) : \mathcal{M}^*_{A,k}(M,J) \to M^k$$

is defined by

$$\mathrm{ev}_J^k(u, z_1, \dots, z_k) = (u(z_1), \dots, u(z_k)).$$

The moduli space $\mathcal{M}^*_{A,k}(M, J)$ is usually not compact but it can be compactified by adding stable maps. A **stable** J-holomorphic map with k marked distinct points

$$u : (C, z_1, \dots, z_k) \to M$$

is a tree $C = \bigcup u_\alpha$ of J-holomorphic maps $u_\alpha : \mathbb{CP}^1 \to M$ with at worst nodal singularities such that if a component $u_\alpha : \mathbb{CP}^1 \to M$ is constant the number of marked and singular points that it contains is greater or equal to three. This implies that the automorphism group of u is finite. The degree of u is defined as

$$\deg u = \sum_\alpha \deg u_\alpha \in H_2(M; \mathbb{Z}).$$

For $A \in H_2(M; \mathbb{Z})$, we denote by $\overline{\mathcal{M}}_{A,k}(M, J)$ the compactified moduli space of J-holomorphic stable maps of degree A with k marked points. The moduli space $\overline{\mathcal{M}}_{A,k}(M, J)$ carries a virtual fundamental class $[\overline{\mathcal{M}}_{A,k}(M, J)]_{\mathrm{virt}} \in H_*(\overline{\mathcal{M}}_{A,k}(M, J), \mathbb{Q})$ that is used for defining the Gromov-Witten invariants [24, 37].

Theorem 2.4.1. *For a generic almost complex structure J the moduli space $\overline{\mathcal{M}}_{A,k}(M, J)$ carries a homology class $[\overline{\mathcal{M}}_{A,k}(M, J)]_{\mathrm{virt}} \in H_*(\overline{\mathcal{M}}_{A,k}(M, J); \mathbb{Q})$.*

The pushforward of $[\overline{\mathcal{M}}_{A,k}(M, J)]_{\mathrm{virt}}$ under $\mathrm{ev}_J^k : \overline{\mathcal{M}}_{A,k}(M, J) \to M^k$ defines a homology class

$$GW_{A,k}(M) \in H_{\dim}(M^k; \mathbb{Q})$$

in dimension

$$\dim \overline{\mathcal{M}}_{A,k}(M, J) = \dim M + 2c_1(TM)(A) + 2k - 6.$$

The class $GW_{A,k}(M)$ is invariant under smooth deformation of (ω, J) through compatible structures and it is called the **Gromov-Witten cycle** *of (M, ω).*

For $\alpha_1, \dots, \alpha_k \in H^*(M)$, the **Gromov-Witten invariant** *is defined as*

$$GW_{A,k}(\alpha_1, \dots, \alpha_k) := \langle \alpha_1 \times \dots \times \alpha_k, GW_{A,k}(M) \rangle = \int_{[\overline{\mathcal{M}}_{A,k}(M,J)]_{\mathrm{virt}}} \mathrm{ev}_k^* \alpha_1 \cup \dots \cup \mathrm{ev}_k^* \alpha_k.$$

If we fix geometric representative $A_i \subset M$ for the Pointcaré duals of each cohomology class α_i and assume that

$$\dim \overline{\mathcal{M}}_{A,k}(M, J) = \dim M + 2c_1(TM)(A) + 2k - 6 = \sum_i \deg \alpha_i.$$

For a generic almost complex structure J, the Gromov-Witten invariant $GW_{A,k}(\alpha_1, \dots, \alpha_k)$ can be interpreted, with appropriate sign, as the number of J-holomorphic perturbed maps of degree A with k marked points $u : (\mathbb{CP}^1, z_1, \dots, z_k) \to M$ such that $u(z_i) \in A_i, i = 1, \dots, k$.

The following statement shows the relationship between J-holomorphic curves and the Gromov width of symplectic manifolds:

Theorem 2.4.2. [7, Theorem 2.2.1] *Let (M^{2n}, ω) be a compact symplectic manifold, and $A \in H_2(M; \mathbb{Z}) \setminus \{0\}$ a second homology class. Suppose that for a dense subset of smooth ω-compatible almost complex structures, the evaluation map*

$$\mathrm{ev}_J^1 : \mathcal{M}^*_{A,1}(M, J) \to M$$

is onto. Then for any symplectic embedding $B^{2n}(r) \hookrightarrow M$, we have

$$\pi r^2 \leq \omega(A),$$

where $\omega(A)$ denotes the symplectic area of A. In particular,

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A).$$

According to Theorem 2.4.2, in order to find an upper bound for the Gromov width of a symplectic manifold (M, ω), we want to prove that for generic almost complex structure $J \in \mathcal{J}(M, \omega)$, the evaluation map:

$$\mathrm{ev}_J^1 : \mathcal{M}^*_{A,1}(M, J) \to M$$

is onto. One way to show ontoness of the evaluation is for example by proving that a Gromov-Witten invariant with one of its constraints being a point is different from zero.

To ensure that the moduli space $\mathcal{M}^*_{A,1}(M, J)$ is a smooth manifold, it suffices to find a **regular** almost complex structure J because being regular is a generic property of J. We quote the construction from [32]. To define regularity, we start with the moduli space $\mathcal{M}^*(A, J)$ of equivalence classes of J-holomorphic curves.

The moduli space $\mathcal{M}^*(A, J)$ can be interpreted as the zero set of a section of an infinite dimensional vector bundle as follows. Let $\mathcal{B} \subset C^\infty(\mathbb{CP}^1, M)$ denote the space of all smooth maps $u : \mathbb{CP}^1 \to M$ that represent the homology class $A \in H_2(M; \mathbb{Z})$. This space can be thought of as an infinite dimensional manifold whose tangent space at $u \in \mathcal{B}$ is the space

$$T_u\mathcal{B} = \Omega^0(\mathbb{CP}^1, u^*TM)$$

of smooth vector fields $\xi(z) \in T_{u(z)}M$ along u. Consider the infinite dimensioal vector bundle $\mathcal{E} \to \mathcal{B}$ whose fiber at u is the space

$$\mathcal{E}_u = \Omega^{0,1}(\mathbb{CP}^1, u^*TM)$$

of smooth J-antilinear 1-forms on $\mathbb{CP}^1$ with values in u^*TM. Then the antilinear part of du defines a section $\mathcal{S} : \mathcal{B} \to \mathcal{E}$ of this vector bundle:

$$\mathcal{S}(u) := (u, \bar{\partial}_J(u)).$$

The moduli space $\mathcal{M}^*(A, J)$ is the zero set of this section. For every $u \in \mathcal{M}^*(A, J)$, we denote by

$$D_u := D_{J,u} := D\mathcal{S}(u) : \Omega^0(\mathbb{CP}^1, u^*TM) \to \Omega^{0,1}(\mathbb{CP}^1, u^*TM)$$

the composition of differential $d\mathcal{S}(u) : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E}$ with the projection

$$\pi_u : T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u.$$

When the operator D_u is onto, $\mathcal{S}$ is traverse to the zero section. Hence, the moduli spaces $\mathcal{M}^*(A, J)$ are smooth finite dimensional manifolds with appropriate dimensions as in Theorem 2.4.1.

2.5 Computation of first Chern numbers using localization

Suppose that a $2n$-dimensional manifold (M, ω) equipped with an almost complex structure J and J can be thought of as multiplication by i. Thus, it makes the tangent bundle TM of M into a complex vector bundle of dimension n. The form ω is said to **tame** J if

$$\omega(v, Jv) > 0$$

for all nonzero $v \in TM$. Different choices of J in the set $\mathcal{J}_\tau(M, \omega)$ of almost complex structures tamed by ω give rise to isomorphic complex vector bundle (TM, J). Thus the **Chern classes** of these bundles are independent of the choice of J and will be denoted by $c_i(TM)$ [32, section 1.1]. For the rest of the thesis, we shall only use the first Chern class, and what will be relevant is the value

$$c_1(A) := \langle c_1(TM), A \rangle$$

which takes on a homology class $A \in H_2(M)$.

If A is represented by a smooth map $u : \mathbb{CP}^1 \to M$ then $c_1(A) = c_1(E)$ is the first Chern number of the pullback tangent bundle $E := u^*TM$. But every complex bundle E over $\mathbb{CP}^1$ decomposes as a sum of complex line bundles $E = L_1 \oplus \ldots \oplus L_n$. Correspondingly

$$c_1(E) = \sum_i c_1(L_i).$$

Since the first Chern number of a complex line bundle is the same as its Euler number, we can calculate the $c_1(L_1)$ directly.

Example 2.5.1. If A is the class of the sphere $S = \mathrm{pt} \times S^2$ in $M = V \times S^2$ then

$$TM|_S = TS \oplus L_2 \oplus \ldots \oplus L_n,$$

where the line bundles L_k are trivial. It follows that

$$c_1(A) = c_1(TM|_S) = c_1(TS) = \chi(S) = 2$$

where $\chi(S)$ is the Euler characteristic of S.

The following lemma provides a criterion to determine whether or not a J-holomorphic curve is regular as described in the section above.

Lemma 2.5.2 ([32], Lemma 3.3.1)**.** *Let M be a compact, closed and connected symplectic manifold and J be an almost complex structure on M. Assume J is integrable and let $u : \mathbb{CP}^1 \to M$ be a J-holomorphic sphere. If every summand of u^*TM has Chern number $c_1 \geq -1$, then D_u is onto.*

An almost complex structure J is **regular** for a homology class $A \in H_2(M; \mathbb{Z})$ if A has a J-holomorphic representative $u : \mathbb{CP}^1 \to M$ such that D_u is onto. We know that $TM|_{u(\mathbb{CP}^1)}$ splits as a sum of line bundles L_i and for each summand L_i in the sum, by naturality of Chern classes,

$$u^*(c_1(L_i)) = c_1(u^*(L_i)).$$

So we have

$$c_1(u^*(L_i))[\mathbb{CP}^1] = c_1(L_i)[u(\mathbb{CP}^1)] = c_1(L)[A].$$

Suppose that a compact Lie group G acts on a compact, closed, connected and oriented manifold M. Let $EG \to BG$ denote the classifying bundle for G. The equivariant cohomology ring

$$H_G^*(M; R) := H^*(M \times_G EG; R),$$

with coefficients in a ring R, encodes topological information about the manifold and the action.

When given a Hamiltonian action on a symplectic manifold, a variety of techniques has made computation of $H_G^*(M; R)$ tractable. The work of Goresky-Kottwitz-MacPherson [11] describes this ring combinatorially when G is a torus, R a field and the action has a specific form. A theorem of Kirwan [23] states that the inclusion of the fixed points induces an injective map in equivariant cohomology. When M is equipped with a Hamiltonian action of a compact torus $\mathbb{T}$, the localization theorem relates the integral, or push-forward to a point, of an equivariant cohomology class on the whole manifold M to the sum of integrals of the class over each component of the fixed point set, corrected with an equivariant characteristic class of the normal bundle to the fixed point component.

Let $\pi : M \to pt$ be a constant map from M to a point, and $\pi^F : F \to pt$ the same for a connected component of the fixed set M^T. The normal bundle to a component F of the fixed point set M^T, denoted as $\nu(F \subset M)$ is an **equivariant vector bundle** which can be associated with **equivariant characteristic classes** [13, Appendix C]. The equivariant characteristic classes live in the equivariant cohomology of the base space. The bundle $\nu(F \subset M)$ is a subbundle of the tangent bundle to M, restricted to F; that is

$$\nu(F \subset M) \subset TM|_F.$$

Now we introduce the push-forward version of the localization.

Theorem 2.5.3 (ABBV Localization, [2, 3])**.** *Suppose a compact torus T acts on a compact manifold M. Then for any class $u \in H_T^*(M;\mathbb{C})$,*

$$\pi_*(u) = \sum_{F \subseteq M^T} \pi_*^F\left(\frac{u|_F}{e_T(\nu(F \subseteq M))}\right), \tag{2.2}$$

where the sum on the right-hand side is taken over all connected components F of the fixed point set M^T, and $u|_F$ is the restriction of u to F.

The formula 2.2 simplifies greatly in the case M^T consists of isolated fixed points. In this case, the normal bundle $\nu(F \subseteq M)$ is just the tangent space to the fixed point:

$$\nu(F \subseteq M)) = TM|_F = T_F M,$$

and the equivariant Euler class is the product of the isotropy weights for the T-action on this tangent space $T_F M$. In this case of push-forward to a point, we may think of π_* as an equivariant integral, using the Cartan model of differential forms. The next example demonstrates how we compute the first Chern number of a spherical homology class in $H^2(M, \mathbb{R})$.

Example 2.5.4. We revisit the 2-dimensional complex projective plane $(\mathbb{CP}^2, \omega_{FS})$ equipped with the $\mathbb{T}^2$-action described in Example 2.1. Let S denote the 2-sphere whose moment image corresponding to the edge connecting $(0,0)$ and $(\frac{1}{2}, 0)$. Then S has the form:

$$S = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid z_2 = 0\}.$$

Let A denote the second homology class corresponding to S. We apply Theorem 2.5.3 to compute the first Chern number $c_1(A) = \langle c_1(T\mathbb{CP}^2), A\rangle$.

To do so, we consider the equivariant first Chern class $c_1^{\mathbb{T}^2}(T\mathbb{CP}^2) \in H^*_{\mathbb{T}^2}(M;\mathbb{C})$ and $\pi_*(c_1^{\mathbb{T}^2}(T\mathbb{CP}^2)) = \int_S c_1^{\mathbb{T}^2}(T\mathbb{CP}^2)$. By naturality of equivariant cohomology, we also have that

$$c_1(A) = \int_S c_1^{\mathbb{T}^2}(T\mathbb{CP}^2).$$

By Theorem 2.5.3,

$$c_1(A) = \sum_{F \subseteq M^T} \pi_*^F\left(\frac{u|_F}{e_T(\nu(F \subseteq M))}\right).$$

Let p and q be elements of $\mathbb{CP}^2$ corresponding to fixed points $(0,0)$ and $(\frac{1}{2},0)$, respectively. The isotropy weights at $(0,0)$ are x and y, and the isotropy weights at $(\frac{1}{2},0)$ are $-x$ and $-x+y$ as shown in Figure 2.6. Here S contains two fixed points so we have a sum of two terms, each term corresponds to a fixed point. Since the equivariant Euler characteristic class is the product of isotropy weights along the tangent space of S^2 at the fixed points while the first equivariant Chern class is the sum of isotropy weights at the fixed points. Then at $(0,0)$, $e_{\mathbb{T}^2}(T_p\mathbb{CP}^2) = x$ and $c_1^{\mathbb{T}^2}(T_p\mathbb{CP}^2) = x+y$ while at $(\frac{1}{2},0)$ we have $e_{\mathbb{T}^2}(T_q\mathbb{CP}^2) = -x$ and $c_1^{\mathbb{T}^2}(T_q\mathbb{CP}^2) = -2x+y$. Thus, we have that

$$c_1(A) = \frac{x+y}{x} + \frac{-2x+y}{-x} = \frac{3x}{x} = 3.$$

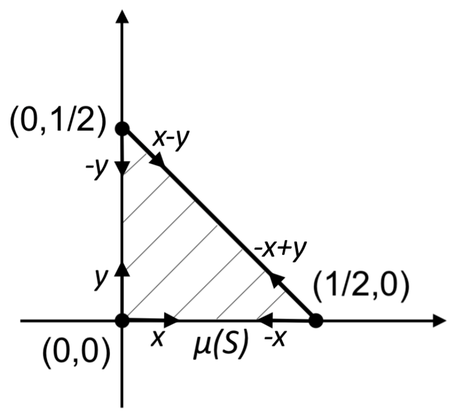


Figure 2.6: Moment polytope of $\mathbb{CP}^2$ with labeled isotropy weights

CHAPTER 3

FOUR DIMENSIONAL TORIC SYMPLECTIC MANIFOLDS

3.1 Introduction

In this chapter, we suppose that M is a compact, closed and connected 4-dimensional toric symplectic manifold equipped with an effective Hamiltonian 2-torus $\mathbb{T}^2$. We consider a circle subgroup S^1 of $\mathbb{T}^2$ such that the symplectic cut by the circle subgroup at a regular value will give two smooth toric symplectic manifolds. We call this operation **a smooth cutting**. Under a smooth cutting, the moment image of the resulting manifold is also obtained from the moment image of the original manifold by the hyperplane defined by the weight of the circle subgroup. We demonstrate this phenomenon in the next example.

Example 3.1.1. Consider the symplectic manifold $M = S^2 \times S^2$ with the standard symplectic form $\omega \oplus \omega$ where ω is the standard symplectic form on S^2. The standard torus $\mathbb{T} = S^1 \times S^1$ acts on M by the standard action of S^1 on S^2. This action comes with a moment map whose moment image is a square as in Figure 3.1 below. The diagonal circle

$$D = \{(a, a) \in \mathbb{T} \mid a \in S^1\}$$

acts Hamiltonianly on M and we can apply symplectic cutting by this circle action. See the image of the resulting moment polytope in the Figure 4.1 below.

In this chapter, we primarily investigate the question whether or not the Gromov width of the resulting symplectic manifold is at most the Gromov width of the orginal four-dimensional toric symplectic manifold. With that being said,

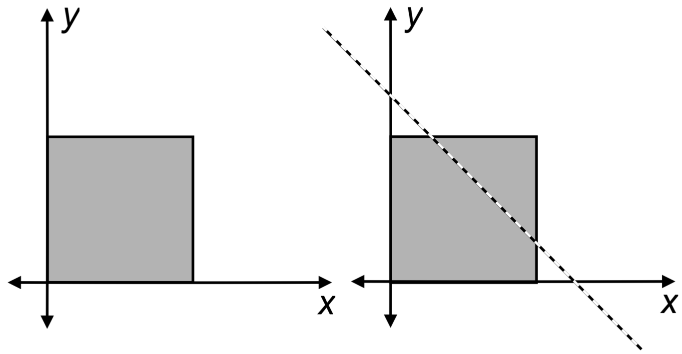


Figure 3.1: The shaded square on the left is the moment polytope of $M = S^2 \times S^2$ and the moment polytopes of symplectic cuts of M are sliced by the diagonal line.

we will begin to explore the concept of toric domains whose moment polytope is highly related to the moment polytope of our manifold of interest.

Definition 3.1.2. A **toric domain** M_Ω is the preimage of a closed region $\Omega \subset \mathbb{R}^2$ in the first quadrant under the map $\mu : \mathbb{C}^2 \to \mathbb{R}^2$ where $\mu(z_1, z_2) = (\pi|z_1|^2, \pi|z_2|^2)$.

Example 3.1.3. We have the following two examples:

i) An ellipsoid

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}$$

is a toric domain whose moment polytope is a right triangle with legs on the axes.

ii) A polydisk is toric domain whose moment polytope is a rectangle with two sides on the axes.

Definition 3.1.4. A **convex (concave)** toric domain is a toric domain X_Ω where Ω is a closed region in the first quadrant bounded by the axes and a convex (concave) curve from $(a, 0)$ to $(b, 0)$ for a and b positive real numbers.

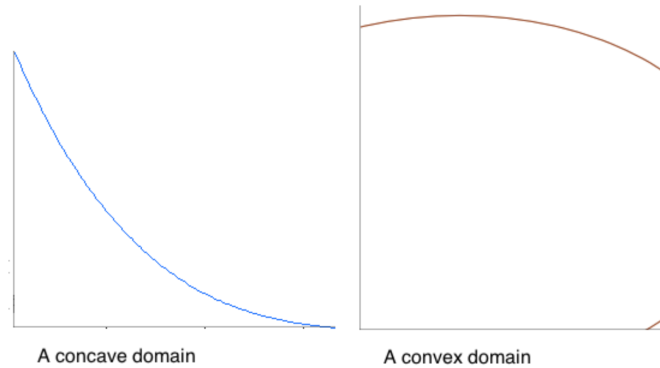


Figure 3.2: Examples of concave and convex domains.

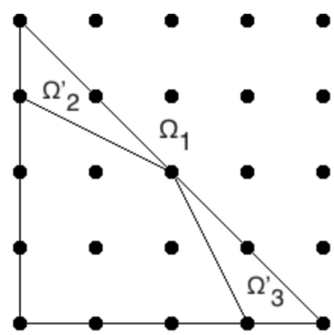


Figure 3.3: The inductive decomposition of a convex toric domain.

Example 3.1.5. Ellipsoids are both concave and convex toric domains because their moment polytopes are triangles.

We now quote the decomposition of the weight sequence of a convex toric domain from Cristofaro-Gardiner [8]. We suggest to consult Figure 3.3 closely as we describe the decomposition. If Ω is a triangle with vertices $(0, 0)$, $(0, b)$ and $(b, 0)$ then the weight sequence of Ω is (b). Otherwise, let $b > 0$ be the smallest real number such that Ω is contained in the triangle with vertices $(0, 0)$, $(0, b)$ and $(b, 0)$. Call this triangle Ω_1. The line $x + y = b$ intersects the upper boundary of Ω in a line segment from $(x_1, b - x_1)$ to $(x_2, b - x_2)$ with $x_1 \leq x_2$. Let Ω'_2 denote the closure of the portion of $\Omega_1 \setminus \Omega$ that is to the left of x_1 and below the line $x + y = b$, and let Ω_3' denote the closure of the portion of $\Omega_1 \setminus \Omega$ that is below $b - x_2$ and below the line $x + y = b$.

The key point is now that Ω_2' and Ω_3' are both affine equivalent to concave toric domains. We then define

$$\omega(\Omega_1) = (b; \omega(\Omega_2') \cup \omega(\Omega_3')).$$

Thus, the weight sequence for a convex toric domain consists of a number, and then a sequence of numbers. We call the first number in this sequence the **head**, and we call the other numbers the **negative weight sequence**.

3.2 Embedded contact homology capacities

In this section, we briefly review embedded contact homology (ECH) capacities. Hutchings developed ECH capacities which is used to provide obstructions to the symplectic embedding of a ball into four-dimensional symplectic manifolds [18]. This technique is crucial in determining the upper bound of the Gromov width of a symplectic 4-dimensional manifold.

Definition 3.2.1. Let (M, ω) be a symplectic four-manifold, possibly with boundary or corners, non-compact, and/or disconnect. Its ECH capacities are a sequence of real numbers

$$0 = c_0(M, \omega) \leq c_1(M, \omega) \leq c_2(M, \omega) \leq \ldots \leq \infty$$

satisfies the following properties:

(Monotonicity) If there exists a symplectic embedding $(M, \omega) \to (M', \omega')$, then

$c_k(M,\omega) \leq c_k(M',\omega')$ for all k.

(Conformality) If $r > 0$, then

$$c_k(M, r\omega) = rc_k(M,\omega).$$

(Disjoint union)

$$c_k\left(\bigsqcup_{i=1}^{n}(M_i,\omega_i)\right) = \max_{k_1+\ldots+k_n=k}\sum_{i=1}^{n} c_{k_i}(M_i,\omega_i).$$

(Ellipsoid) If $a, b > 0$, $c_k(E(a,b)) = N(a,b)_k$, where $N(a,b)$ defines as follows: define $(a,b)_k$ to be the k^{th} smallest entry int he array $(am+bn)_{m,n\in\mathbb{N}}$, counted with repetitions. Denote the sequence $((a,b)_{k+1})_{k\geq 0}$ by $N(a,b)$.

For example, $N(1,2) = (0,1,2,2,3,3,4,4,4,5,5,5,\ldots)$. McDuff [31] shows that there exists a symplectic embedding of $\mathrm{int}(E(a,b)) \to E(c,d)$ if and only if $N(a,b)_k \leq N(c,d)_k$ for all k where $\mathrm{int}(E(a,b))$ denotes the interior of $E(a,b)$. The following theorem [8, Theorem A.1] is used in the proof of the Proposition 3.3.2. To state the formula, recall the "sequence subtraction" operation; given two sequences S and T we have

$$(S \mathbin{\boldsymbol{-}} T)_k = \min_{\ell\geq 0}(S_{k+\ell} - T_\ell).$$

Theorem 3.2.2. *Let M_Ω be a convex toric domain, let b be the head of the weight expansion for M_Ω, and let b_i be the i^{th} term in the negative weight expansion for M_Ω. Then*

$$c_{ECH}(M_\Omega) = c_{ECH}(B(b)) \mathbin{\boldsymbol{-}} c_{ECH}(\sqcup_i B(b_i)).$$

There is a standard symplectic form ω on $\mathbb{CP}^2$ such that $\langle L,\omega\rangle = 1$, where L denotes the homology class of a line. With this symplectic form, $\mathrm{vol}(\mathbb{CP}^2) = 1/2$, and there exists a symplectic embedding of the interior of $B(1)$, denoted as $\operatorname{int}(B(1))$, into $\mathbb{CP}^2$. A symplectic form ω on M_k whose cohomology class is given by

$$\mathrm{PD}[\omega] = L - \sum_{i=1}^{m} a_i E_i, \tag{3.1}$$

where E_i denotes the homology class of the i^{th} exceptional divisor, and PD denotes Poincaré duality. The canonical class for this symplectic form (namely - c_1 of the tangent bundle as defined using an ω-compatible almost complex structure) is given by

$$\mathrm{PD}(K) = -3L + \sum_{i=1}^{m} E_i. \tag{3.2}$$

The next proposition is a crucial result regarding to k-fold blowups of $\mathbb{CP}^2$.

Proposition 3.2.3 ([28, Proposition 6])**.** Let $a_1, \ldots, a_m > 0$. The following are equivalent:

(a) There exists a symplectic embedding

$$\bigsqcup_{i=1}^{m} B(a_i) \to \operatorname{int}(B(1)).$$

(b) There exists a symplectic form ω on M_k satisfying (3.1) and (3.2).

3.3 Gromov width of four dimensional toric symplectic manifolds

Now we turn our attention to the main result of this chapter. We show that the Gromov width of any smooth cut of a four dimensional toric manifold is at most the Gromov width of the original manifold. We compare the ECH capacities of the orginal manifold with the ECH capacities of resulting cut via their moment polytopes and the associated moment polytopes.

Definition 3.3.1. Let M be a 4-dimensional toric symplectic manifold with a moment polytope Ω. A **convex toric domain** M_Ω associated with M is the preimage of Ω under the moment map $\phi : \mathbb{C}^n \to \mathbb{R}^2$ where $\phi(z_1, z_2) = (\pi|z_1|^2, \pi|z_2|^2)$.

Proposition 3.3.2. *Let M be a 4-dimensional toric symplectic manifold and M_Ω be a convex toric domain associated with M. Then there exists a ball of radius a embedded into M if and only if there exists a ball of radius a embedded into M_Ω.*

Proof. If $B(a)$ embeds into M_Ω, then $B(a)$ embeds into M because M contains M_Ω as an open dense subset. Thus, if there exists a ball of capacity a embedded in M_Ω then there exists a ball of capacity a embedded in M.

To complete the proof, suppose that $B(a)$ is the largest ball that could be embedded into M_Ω and suppose that $B(a+c)$ embeds into M. We want to show that $c = 0$. Since $B(a+c)$ is a ball in M, we can blow up this ball to obtain a blow-up of M, called M'. By Proposition 3.2.3, this blow-up is equivalent to a ball packing of open balls. Namely, $B(a+c) \cup (\bigsqcup_i B(b_i)) \hookrightarrow B(b)$ where b is the capacity of M. This implies that $c_k(B(a+c) \leq c_k(B(b)) - c_k(\sqcup B(b_i))$ for all k. By Theorem 3.2.2, $c_k(M_\Omega) = c_k(B(b)) - c_k(\sqcup B(b_i)), \forall k$. From this, we can conclude that $c_k(B(a+c)) \leq c_k(M_\Omega)$. Then by monotonicity, $B(a+c) \hookrightarrow M_\Omega$ but $B(a)$ is the largest ball could be embedded into M_Ω so this implies $c = 0$. $\square$

Theorem 3.3.3. *Let M be a 4-dimensional toric symplectic manifold and $\widetilde{M}$ be a smooth cut of M. Then*

$$\operatorname{Gwidth}(\widetilde{M}) \leq \operatorname{Gwidth}(M).$$

Proof. Let M_Ω be the convex toric domain associated with M and $\widetilde{M}_{\widetilde{\Omega}}$ be the convex toric domain associated with $\widetilde{M}$ so that $\widetilde{\Omega} \subset \Omega$. This implies that $\widetilde{M}_{\widetilde{\Omega}} \subset$

M_Ω. Let $\epsilon > 0$ and $\mathrm{Gwidth}(\widetilde{M}) = \pi a^2$. Then $B(a-\epsilon)$ embeds into $\widetilde{M}$. By Proposition 3.3.2, $B(a-\epsilon)$ embeds into $\widetilde{M}_{\widetilde{\Omega}}$, which implies that $B(a-\epsilon)$ embeds into M_Ω because $\widetilde{M}_{\widetilde{\Omega}} \subset M_\Omega$. Again by Proposition 3.3.2, $B(a-\epsilon)$ embeds into M. Since ϵ is arbitrary,

$$\mathrm{Gwidth}(\widetilde{M}) = \pi a^2 \leq \mathrm{Gwidth}(M).$$

$\square$

We discuss an application of our main result to k-fold blow-ups of a four dimensional toric manifold. A blow up of a manifold is a special case of a symplectic cut and it is defined in Remark 2.1.2. The following theorem is a well-known classification of four-dimensional toric manifolds.

Theorem 3.3.4. *[35, Theorem 1.28] Any four-dimensional toric manifold is obtained by blowing up $\mathbb{CP}^2$ or $S^2 \times S^2$ finitely many times.*

Symplectic k-fold blow-ups of $\mathbb{CP}^2$ has been studied extensively by Karshon, Kessler and Pinsonnault [19, 20, 21]. Let M be a four dimensional toric manifold. By Theorem 3.3.4, M is k-fold blow-ups of either $\mathbb{CP}^2$ or $S^2 \times S^2$. We denote M_i the manifold obtained by each blow up where M_1 is either $\mathbb{CP}^2$ or $S^2 \times S^2$ and $M_k = M$. Then by Theorem 3.3.3, we have the following inequalities:

$$\mathrm{Gwidth}(M_1) \geq \mathrm{Gwidth}(M_2) \geq \ldots \geq \mathrm{Gwidth}(M_k).$$

Our main theorem ensures that there is a monotonicity property of the Gromov width that k-fold blow-ups must obey.

CHAPTER 4

GRASSMANNIAN MANIFOLDS

4.1 Introduction

In this chapter we compute the Gromov width of symplectic cuts of the complex Grassmannian manifold $\mathcal{G}r_{\mathbb{C}}(k,n)$ by a circle weight. The result provides evidence that the Gromov width of a symplectic cut of a Grassmannian manifold is at most the Gromov width of the original manifold.

This chapter is organized as follows: in the second section, we review background information on the Grassmannian manifold $\mathcal{G}r_{\mathbb{C}}(k,n)$ of k-planes in $\mathbb{C}^n$ and describe the moment polytope of the action of a maximal torus of $U(n)$ on $\mathcal{G}r_{\mathbb{C}}(k,n)$. In the third section, we describe the slices of the moment polytope. In the fourth section, we classify smooth cuts on $\mathcal{G}r_{\mathbb{C}}(k,n)$. In the fifth section, we compute a lower bound and an upper bound for the Gromov width of symplectic cuts of $\mathcal{G}r_{\mathbb{C}}(k,n)$ by circle weights. As the bounds agree, this determines the Gromov width of the cuts.

4.2 Background on Grassmannian

We let M denote the Grassmannian $\mathcal{G}r_{\mathbb{C}}(k,n)$ of complex k-planes in $\mathbb{C}^n$. This manifold is a homogenous space: it has a transitive action of $U(n)$. It is also a coadjoint orbit of $U(n)$. Coadjoint orbits of $U(n)$ can be identified with a set of isospectral Hermitian matrices $\mathcal{H}_\lambda$. For the Grassmannian, we may choose k eigenvalues to be equal to $\dfrac{-1}{k}$ and the remaining $(n-k)$ eigenvalues to be equal to $\dfrac{1}{n-k}$. The sum of all eigenvalues equals to 0. For the rest of the chapter, we will identify

$$\mathcal{G}r_{\mathbb{C}}(k,n) \cong H_\lambda \ \text{ where } \ \lambda = \left(\frac{-1}{k}, \ldots, \frac{-1}{k}, \frac{1}{n-k}, \ldots, \frac{1}{n-k}\right)$$

and the explicit identification is described in section 4.3. Kostant, Kirillov, and Souriau proved that coadjoint orbits of compact, connected Lie groups may be equipped with a natural symplectic form. The coadjoint action is Hamiltonian with moment map the inclusion of $\mathcal{H}_\lambda \cong \mathcal{G}r_{\mathbb{C}}(k,n) \hookrightarrow \mathfrak{u}(n)^*$. A choice of maximal torus $T \hookrightarrow U(n)$ induces a map on the duals of the Lie algebras $\mathfrak{u}(n)^* \to \mathfrak{t}^*$. The coadjoint action of T on M has moment map the composition

$$\mu : M \cong \mathcal{H}_\lambda \to \mathfrak{u}(n)^* \to \mathfrak{t}^*.$$

We also identify $\mathfrak{t}^*$ with $\mathbb{R}^n$. The identifications above make $\mu(A) = \operatorname{diag}(A) = (a_{11}, \ldots, a_{nn})$ the diagonal entries of the matrix $A \in \mathcal{H}_\lambda$. The Atiyah/Guillmeimin-Sternberg convexity Theorem ([1], [14]) guarantees that $\mu(M)$ is a convex polytope.

Writing H_i for the diagonal matrix $E_{i,i}$ with a single 1 in the (i,i) position and 0's elsewhere, we have the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{su}(n)$

$$\mathfrak{h} = \{a_1H_1 + a_2H_2 + \ldots + a_nH_n \mid a_1 + a_2 + \ldots + a_n = 0\}.$$

Note that H_i is not in $\mathfrak{h}$. We can correspondingly write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \ldots, L_n\}/(L_1 + L_2 + \ldots + L_n = 0)$$

where $L_i(H_j) = \delta_{i,j}$. We often write L_i for the image of L_i in $\mathfrak{h}^*$ and call L_i a **circle weight**. The image of the moment polytope $\mu(M)$ lies in $\mathfrak{h}^*$. The circle weights $L_1, L_2, \ldots, L_n$ can be realized as the lattice generated by the vertices of a regular $(n-1)$-simplex Δ centered at the origin. By this assignment, each vertex of the moment polytope $\mu(H_\lambda)$ is a sum of k elements from the set $\{L_1, \dots, L_n\}$ (see [10], Section 15.2 for more details). Note that the number of vertices of $\mu(H_\lambda)$ is $\binom{n}{k}$ and the degree of each vertex is $k(n-k)$.

Example 4.2.1. Figure 4.1 shows a moment polytope Δ of the Grassmannian $\mathcal{G}r(2,4)$ of 2-planes in $\mathbb{C}^4$ under the action of the standard maximal torus. The circle weights L_1, L_2, L_3, and L_4 are vertices of a regular 3-simplex centered at the origin in $\mathbb{R}^3$ and the vertices of Δ are labeled $L_1+L_2, L_1+L_3, L_1+L_4, L_2+L_3, L_2+L_4$, and L_3+L_4 in the figure.

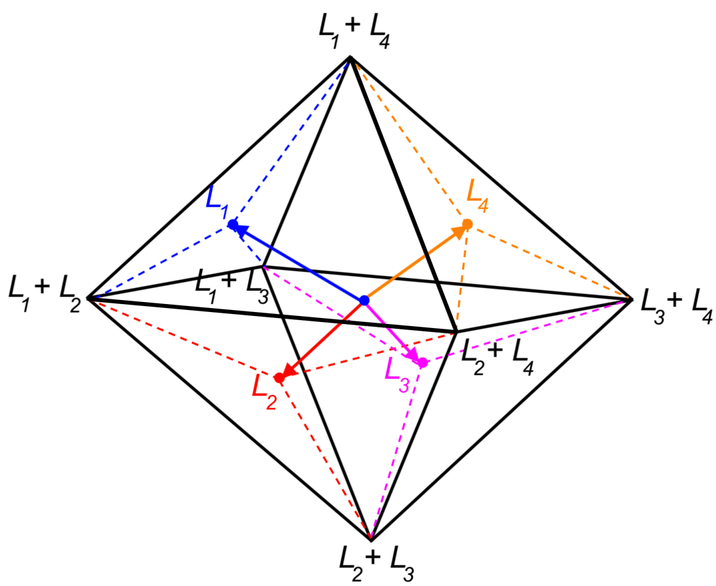


Figure 4.1: Moment polytope of $\mathcal{G}r_{\mathbb{C}}(2,4)$ with labeled weights

The circle weight L_i defines an inclusion $i : S^1 \hookrightarrow T$ by $i(\theta) = (\theta, 1, \dots, 1)$ for all $\theta \in S^1$. This inclusion defines a Hamiltonian circle action on M and induces a momentum map $f : H_\lambda \to \mathbb{R}$ by $f(A) = a_{ii}$ for all $A \in H_\lambda$. The image of f is the closed interval $\left[-\frac{1}{k}, \frac{1}{n-k}\right]$. For any $0 < \epsilon < \frac{n}{k(n-k)}$, a symplectic cut at the regular value $\frac{1}{n-k} - \epsilon$ gives us two symplectic cuts $M_{(-\infty, \frac{1}{n-k}-\epsilon]}$ and

$M_{\left[\frac{1}{n-k}-\epsilon,\infty\right)}$. These two spaces are symplectic manifolds because the circle action is free at the regular value $\dfrac{1}{n-k}-\epsilon$. This fact is proved in Proposition 4.5.1.

It is known that dimension of $H_2(M;\mathbb{Z})$ is 1. Let $A \in H_2(M;\mathbb{Z})$ be a generator. The pre-image of any edge in the moment polytope under μ is a sphere that represents the second homology class A. Karshon and Tolman proved that the Gromov width of Grassmannian $\mathcal{G}r(k,n)$ is $\omega(A)$ [22, Theorem 1]. In order to prove the lower bound, they construct an explicit symplectic embedding of an equivariant ball of capacity $\omega(A)$ centered at a fixed point. Whether or not these equivariant balls survive the symplectic cutting procedure depends on the choices made when making the cut. In order to study the Gromov width of symplectic cuts of the Grassmannian manifolds, we classify symplectic cutting into two types: pervasive and non-pervasive.

Definition 4.2.2. Let (M,ω) be a $2n$-dimensional symplectic manifold. A symplectic cut is called **pervasive** if for every vertex p in the moment polytope Δ of M, there is an edge of p incident to the slice. Otherwise, we call the symplectic cut **non-pervasive**.

Our goal is to find obstructions to the symplectic embedding of a ball of a certain capacity into M. We utilize the theory of J-holomorphic curves, so we first classify the J-holomorphic curves in the class A.

Let (ω_λ, J) be the Kähler structure of $H_\lambda \cong \mathcal{G}r(k,n)$ where ω_λ is the pull back of Kirillov-Kostant-Souriau form defined on the coadjoint orbit $\mathcal{G}r_{\mathbb{C}}(k,n)$ of $\mathfrak{u}(n)^*$ via the identification

$$H_\lambda \cong \mathcal{G}r_{\mathbb{C}}(k,n).$$

The complex structure J is coming from the presentation of H_λ as a quotient of complex Lie groups $Sl(n,\mathbb{C})/P$, where $P \subset Sl(n,\mathbb{C})$ is a parabolic subgroup of block upper triangular matrices. Let

$$\mathcal{M}_A(M,J) = \{u : \mathbb{CP}^1 \to M \mid u \text{ is } J\text{-holomorphic and } u_*[\mathbb{CP}^1] = A\}$$

be the moduli space of J-holomorphic curves of class A defined on M. The elements of this moduli space are called **holomorphic lines** in the class A of the Grassmannian manifold M.

For a holomorphic curve $u : \mathbb{CP}^1 \to M$ in the class A, we define the **kernel** of u as the intersection of all the k-dimensional subspaces $V \subset \mathbb{C}^n$ that are in the image of u. Likewise, the **span** of u is the linear span of these subspaces. That is

$$\ker(u) = \bigcap_{V \in u(\mathbb{CP}^1)} V \qquad \text{and} \qquad \operatorname{span}(u) = \sum_{V \in u(\mathbb{CP}^1)} V.$$

The kernel and span of u are of dimension $k-1$ and $k+1$ [4, Lemma 1], respectively; and they determine the holomorphic line up to parametrization. In other words, if there is a holomorphic line $v : \mathbb{CP}^1 \to M$ such that $\ker(u) = \ker(v)$ and $\operatorname{span}(u) = \operatorname{span}(v)$, then there exists $(g : \mathbb{CP}^1 \to \mathbb{CP}^1) \in PSL(2;\mathbb{C})$ such that $v = u \circ g$. Moreover, we have $u(\mathbb{CP}^1) = \{V^k \in M \mid \ker(u) \subset V^k \subset \operatorname{span}(u)\} \subset M$ [4, section 2.1].

Thus we may identify $\mathcal{M}_A(M,J)/PSL(2;\mathbb{C}) \simeq Fl(k-1,k+1;n)$, where $Fl(k-1,k+1;n)$ denotes the partial flag manifold of nested complex subspace sequences

$$V^{k-1} \subset V^{k+1} \subset \mathbb{C}^n,$$

a $(k-1)$-dimensional subspace contained in a $(k+1)$-dimensional one. For example with $k=2, n=4$, and fixing $V = (V^1, V^3) \in Fl(1,3;4)$, we will denote by u_V the (unparametrized) holomorphic line

$$\mathbb{CP}^1 \simeq \{V^2 \in M : V^1 \subset V^2 \subset V^3\} \subset M.$$

4.3 Slices of the moment polytope

For a slice S of the moment polytope, we want to describe explicitly $\mu^{-1}(S)$ as a submanifold of M. Note that M is a partial flag manifold $Fl(k;n)$ with flag $V = (V^k \subset \mathbb{C}^n)$. For $\lambda = \left(-\frac{1}{k}, \ldots, -\frac{1}{k}, \frac{1}{n-k}, \ldots, \frac{1}{n-k}\right)$, we can form the Hermitian operator

$$A_\lambda(V) = \frac{1}{n-k} \cdot P + \left(-\frac{1}{k}\right)(I - P)$$

where I is the identity map from $\mathbb{C}^n$ to itself and P is the orthogonal projection of $\mathbb{C}^n$ onto V. The correspondence $V \to A_\lambda(V)$ defines a $Sl(n, \mathbb{C})$-equivariant biholomorphism between M and H_λ, see [7, section 4.1].

Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis for $\mathbb{C}^n$ equipped with the standard Hermitian inner product. Let V be a k-plane in $\mathbb{C}^n$, we may write

$$V = \operatorname{span}(v_1, \ldots, v_k)$$

where $v_j = \sum_{i=1}^n v_{ji} e_i$ for $j = 1 \ldots k$ are an orthonormal basis for V. In coordinates, this means $\sum_{i=1}^n v_{ji}\overline{v_{\ell i}} = 0$ for $j \neq \ell$, $\sum_{i=1}^n |v_{ji}|^2 = 1$ for all j.

With respect to the standard basis of $\mathbb{C}^n$, we can compute the matrix representation of the projection map $P : \mathbb{C}^4 \to V$. The matrix P has the form

$$P = \begin{bmatrix} \sum_{i=1}^{k} v_{i1}\overline{v_{i1}} & \sum_{i=1}^{k} v_{i1}\overline{v_{i2}} & \dots & \sum_{i=1}^{k} v_{i1}\overline{v_{in}} \\ \sum_{i=1}^{k} v_{i2}\overline{v_{i1}} & \sum_{i=1}^{k} v_{i2}\overline{v_{i2}} & \dots & \sum_{i=1}^{k} v_{i2}\overline{v_{in}} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{i=1}^{k} v_{in}\overline{v_{i1}} & \sum_{i=1}^{k} v_{in}\overline{v_{i2}} & \dots & \sum_{i=1}^{k} v_{in}\overline{v_{in}} \end{bmatrix}$$

With $\alpha = \dfrac{n}{k(n-k)}$ and $\beta = \dfrac{1}{k}$, the matrix $A_\lambda(V) = \dfrac{1}{n-k} \cdot P + \left(-\dfrac{1}{k}\right)(I - P)$ has the form

$$A_\lambda(V) = \begin{bmatrix} \alpha\left(\sum_{i=1}^{k} |v_{i1}|^2\right) - \beta & \dots & \dots & \dots \\ \dots & \alpha\left(\sum_{i=2}^{k} |v_{i2}|^2\right) - \beta & \dots & \dots \\ \vdots & \vdots & \dots & \vdots \\ \dots & \dots & \dots & \alpha\left(\sum_{i=1}^{k} |v_{in}|^2\right) - \beta \end{bmatrix} \tag{4.1}$$

Thus, $A_\lambda(V)$ is the Hermitian matrix that corresponds to k-plane V in $\mathbb{C}^n$.

Example 4.3.1. Consider the k-plane $V = \operatorname{span}(e_1, \dots, e_k)$. With $v_{11} = v_{22} = \dots = v_{kk} = 1$ and $v_{ij} = 0$ for $i \neq j$, the corresponding Hermitian matrix of this plane V is a diagonal matrix A with $a_{ii} = 1$ if $i \leq k$ and 0 if $i > k$.

Consider the regular value $\dfrac{1}{n-k} - \epsilon$. The set of Hermitian matrices with moment image $\dfrac{1}{n-k} - \epsilon$ is $A = (a_{ij}) \in H_\lambda$ such that $a_{11} = \dfrac{1}{n-k} - \epsilon$. By equation (4.1),

$$\frac{1}{n-k} - \epsilon = \frac{n}{k(n-k)}\left(\sum_{i=1}^{k} |v_{i1}|^2\right) - \frac{1}{k}.$$

This implies that

$$\sum_{i=1}^{k} |v_{i1}|^2 = 1 - \epsilon\frac{k(n-k)}{n}.$$

Thus,

$$f^{-1}(1-\epsilon) = \left\{ \operatorname{span}(v_1, \ldots, v_k) \mid \sum_{i=1}^{n} |v_{ji}|^2 = 1, \right.$$

$$\left. \sum_{i=1}^{n} v_{ji}\overline{v_{\ell i}} = 0, \sum_{i=1}^{k} |v_{i1}|^2 = 1 - \epsilon \frac{k(n-k)}{n} \right\} \tag{4.2}$$

Note that $\dfrac{1}{n-k}$ and $-\dfrac{1}{k}$ again are not regular values of f.

4.4 Smooth cuts on complex Grassmannians

Generically the resulting cut spaces of a symplectic cut are orbifolds. A **smooth cut** is a symplectic cut for which the resulted cut spaces are symplectic manifolds.

Definition 4.4.1. Let F be an ℓ-dimensional face of the moment polytope of $\mathcal{G}r(k,n)$ with vertices $v_1, \ldots, v_m$. The sum $v_1 + \ldots + v_m$ is called the **center weight** of F.

We first classify some smooth cuts on a complex Grassmannian.

Proposition 4.4.2. *In $\mathcal{G}r(k,n)$, symplectic cuts by the center weight of an $\ell-$dimensional simplex face of the polytope are smooth cuts.*

Proof. Each $\ell-$dimensional simplex face of the polytope is the image of an ℓ-dimensional submanifold of $\mathcal{G}r(k,n)$. A symplectic cut by the center weight of an ℓ-dimensional is equivalent to an equivariant blow-up along the corresponding ℓ-dimensional submanifold. Therefore, the cuts are smooth. $\square$

The following proposition tells us when a cut by the center weight of an ℓ-dimensional face will be a pervasive cut.

Proposition 4.4.3. *A smooth cut by the center weight of an ℓ-dimensional face F of the polytope $\mu(\mathcal{G}r(k,n))$ is pervasive if and only if every vertex in the polytope is connected to one of the vertices in F.*

Proof. Let α be the Gromov width of $\mathcal{G}r(k,n)$. Every fixed point in M has a maximal equivariant ball with capacity α centered about it. Each vertex corresponds to a fixed point so there is a maximal equivariant ball at each vertex. If every vertex in the polytope is connected to one of the vertices in F, then it has an edge incident to the slice. Therefore such cuts must be pervasive.

On the other hand, if the cut is pervasive but there is a vertex that is not connected to any vertices in F, then all edges of the vertex are not incident to the slice. Hence the cut is non-pervasive which contradicts our hypothesis. $\square$

Example 4.4.4. An equivariant blow up at a point (see Figure 4.2) or of an edge in $\mathcal{G}r(2,4)$ is pervasive by the previous proposition.

Proposition 4.4.5. *If $\begin{pmatrix} n \\ k \end{pmatrix} < k(n-k)+\ell+2$, then every smooth cut by the center weight of an ℓ-dimensional face F of the polytope $\mu(\mathcal{G}r(k,n))$ is pervasive.*

Proof. Suppose that $\binom{n}{k} < k(n-k)+\ell+2$. Recall that $\binom{n}{k}$ is the number of vertices of the moment polytope of M while $k(n-k)+1$ is the number of vertices contained in the closure of the image of an equivariant ball in the moment polytope. A smooth cut by the center weight of an ℓ-dimensional face F of the polytope $\mu(\mathcal{G}r(k,n))$ will delete $\ell+1$ vertices from $\binom{n}{k}$ original vertices.

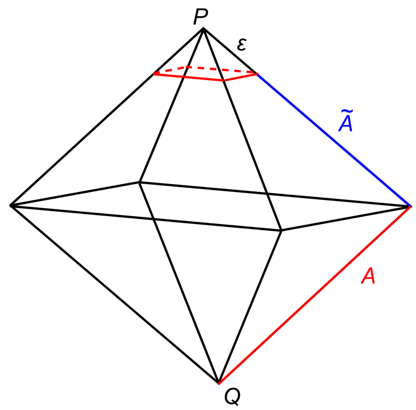


Figure 4.2: Moment polytope of $\mathcal{G}r_{\mathbb{C}}(2,4)$ with a blow-up at the top vertex

Hence, the moment polytope of the resulting cut has $\binom{n}{k} - \ell - 1$ vertices. Since $\binom{n}{k} < k(n-k) + \ell + 2$, this implies $\binom{n}{k} - \ell - 1 < k(n-k) + 1$. This means the total number of vertices in the resulted polytope is strictly less than the number of vertices needed to construct an equivariant ball. This implies that every vertex in the moment polytope has an edge incident to the slice. Hence, the cut is pervasive. $\square$

4.5 Symplectic cuts by circle weight L_i

In order to study Gromov width, we consider symplectic cut spaces that are symplectic manifolds. Fortunately in the case of Grassmannian manifolds the symplectic cuts by the circle weights L_i at a regular value will produce two smooth symplectic cut spaces.

Proposition 4.5.1. *Symplectic cuts on M by the circle weight L_i at a regular value are smooth cuts.*

Proof. Without loss of generality, consider a cut by the circle weight L_1 at a regular value $\dfrac{1}{n-k} - \epsilon$ for some $\epsilon > 0$. At a regular value, smoothness of the cut fails only if there is a nontrivial stabilizer subgroup. Let D be the diagonal circle subgroup

$$D = \{(a, \ldots, a) \in \mathbb{T}^n \mid a \in S^1\}$$

where $\mathbb{T}^n = (S^1)^n$ is the standard torus. The standard torus $\mathbb{T}^n$ acting on $\mathbb{C}^n$ induces an action of $\mathbb{T} = \mathbb{T}^n/D$ on $\mathcal{G}r_k(\mathbb{C}^n)$. This can be identified with the maximal torus in $SU(n)$. Let $\mathrm{Stab}_{\mathbb{T}^n}(V)$ denote the stabilizer subgroup of $\mathbb{T}^n$ of V:

$$\mathrm{Stab}_{\mathbb{T}^n}(V) = \{t \in \mathbb{T}^n \mid t \cdot V = V\}.$$

Thus, it suffices to show that for any $V \in f^{-1}\left(\frac{1}{n-k} - \epsilon\right)$,

$$\mathrm{Stab}_{\mathbb{T}^n}(V) \cap (S^1 \times \{1\} \times \ldots \{1\}) \subset D.$$

We will show that the stabilizer subgroup $\mathrm{Stab}_{\mathbb{T}^n}(V)$ does not contain any element of $S^1 \times \{1\} \times \ldots \{1\}$ except for the identity. Let $t \in S^1$ and represent V by the matrix

$$A = \begin{pmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ v_{k1} & v_{k2} & \ldots & v_{kn} \end{pmatrix}$$

as in (4.1). WLOG we assume $v_{vv} \neq 0$. Under our identifications, we represent $t \cdot V$ by the matrix

$$B = \begin{pmatrix} tv_{11} & v_{12} & \ldots & v_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ tv_{k1} & v_{k2} & \ldots & v_{kn} \end{pmatrix}$$

We apply elementary row operations to both A and B to obtain A' and B' (written in block matrix)

$$A' = \begin{pmatrix} v_{11} & P \\ \mathbf{0} & Q \end{pmatrix}$$

and

$$B' = \begin{pmatrix} tv_{11} & P \\ \mathbf{0} & Q \end{pmatrix}.$$

The only way for A' and B' to be row equivalent is if $tv_{11} = v_{11}$, which only happens when $t = \mathbf{1}$. Hence, $\mathrm{Stab}_{\mathbb{T}^n}(V)$ does not contain any element of $S^1 \times \{1\} \times \ldots \{1\}$ except for the identity. We conclude that

$$\mathrm{Stab}_{\mathbb{T}^n}(V) \cap (S^1 \times \{1\} \times \ldots \{1\}) = \{1\} \subset D.$$

$\square$

Proposition 4.5.2. *Symplectic cuts on M by the circle weight L_i at a regular value are pervasive.*

Proof. Let v be a vertex in the moment polytope Δ of M. Without loss of generality, v is a sum of k weights L_j including L_i. Let w be a vertex having the same sum of k weight L_j but we replace L_i by another L_m that is not already in the weight sum for v. Then vertices v and w are connected by an edge and this edge is incident to the slice defined by the cut. Since v is arbitrary, the symplectic cut is pervasive. $\square$

Let $M_- := M_{\left(-\infty, \frac{1}{n-k} - \epsilon\right]}$ and $M_+ := M_{\left[\frac{1}{n-k} - \epsilon, \infty\right)}$ be the manifolds obtained by the symplectic cut by the circle of weight L_1 at the regular value $\dfrac{1}{n-k} - \epsilon$. Let A be a generator of $H_2(\mathcal{G}r_{\mathbb{C}}(k,n); \mathbb{Z})$ and consider the set

$$S = \{W \in \mathcal{G}r_{\mathbb{C}}(k,n) \mid \mathrm{span}(e_1) \subset W \subset \mathrm{span}(e_1, e_2, e_3)\}.$$

S is the image of a J-holomorphic curve $u : \mathbb{CP}^1 \to M$ in the class A . We denote $A_- \in H_2(M_-;\mathbb{Z})$ and $A_+ \in H_2(M_+;\mathbb{Z})$ the class of the J-holomorphic curve obtained from u after the symplectic cut in M_- and M_+ respectively. For example in the case of $\mathcal{G}r_{\mathbb{C}}(2,4)$, the classes A, A_-, and A_+ are the classes of J-holomorphic curves whose moment map images are shown in Figure 4.3 in M_- and M_+. We will give a lower bound and an upper bound for the Gromov width of M_- that agree, which will verify that the Gromov width of M_- is

$$\omega(A_-) = \frac{n}{k(n-k)} - \epsilon$$

and confirms the monotonicity relation:

$$\mathrm{Gwidth}(M_-) \leq \mathrm{Gwidth}(M).$$

Similarly, the Gromov width of M_+ is ϵ and confirms the monotonicity relation:

$$\mathrm{Gwidth}(M_+) \leq \mathrm{Gwidth}(M).$$

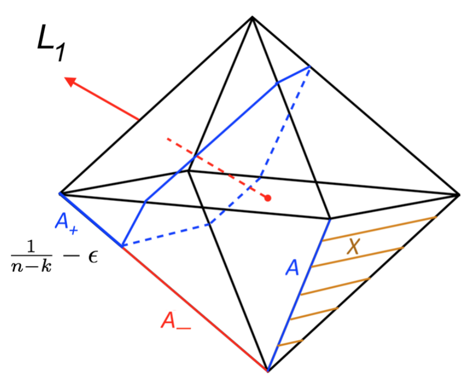


Figure 4.3: Moment polytope of $\mathcal{G}r_{\mathbb{C}}(2,4)$ with symplectic cut along L_1

4.5.1 Lower bound

Proposition 4.5.3. *The Gromov width of M_- is at least $\omega(A_-)$. Similarly, the Gromov width of M_+ is at least $\omega(A_+)$.*

Proof. In the case of M_-, by Proposition 2.3.4, we can embed an equivariant ball B of capacity $\omega(A_-)$ at any fixed point not on the slice. As in Figure 4.4, the image of the equivariant ball B is the polytope in red color centered at the fixed point P. In this case, $\omega(A_-) = \dfrac{n}{k(n-k)} - \epsilon$. By exhibiting such an embedding, we have established $\mathrm{Gwidth}(M_-) \geq \omega(A_-)$. A similarly result holds for M_+. □

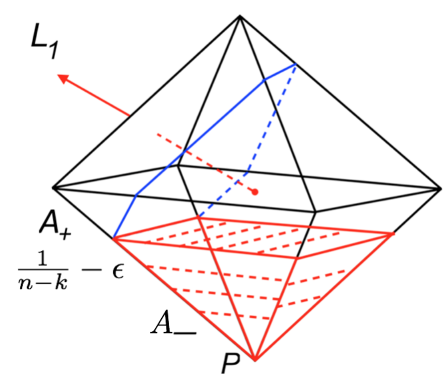


Figure 4.4: Image of an equivariant ball in red color in moment polytope of M_-

4.5.2 Upper bound

In this section we will show that the J_0-holomorphic curves in the class of A_- and A_+ provide an obstruction to symplectic embedding in M_- and M_+ respectively, where J_0 is the standard almost complex structure inherited from M after the smooth cut. Many details in the proof for M_- are similar to those for M_+ but not all. We will carefully describe results for both M_- and M_+. To apply the theory of J-holomorphic curves, we first claim that J_0 is integrable, i.e. it arises from an underlying complex structure on M_- and M_+.

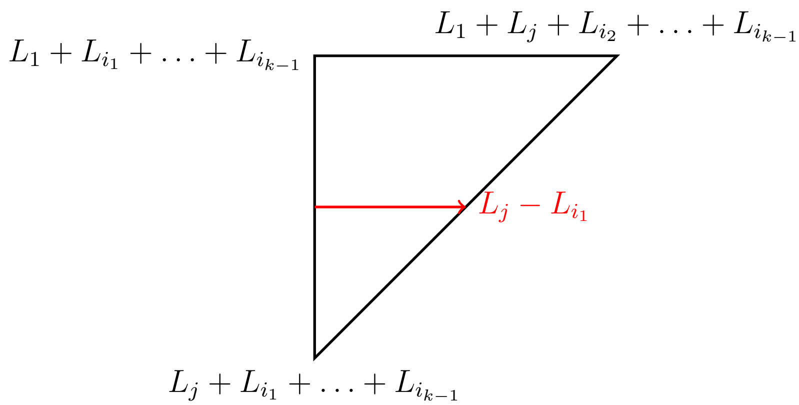
Lemma 4.5.4. *The standard almost complex structure J_0 on M_- and M_+ is integrable.*

Proof. The symplectic cut M_- was obtained from the symplectic quotient of the product of two Kähler manifolds M and $\mathbb{C}$. Since the symplectic quotient of Kähler manifold is again Kähler, $\widetilde{M}$ is a Kähler manifold [25, Remark 1.1]. Hence, the standard almost complex structure J_0 on M_- is integrable. A similar result holds for M_+. $\square$

Lemma 4.5.5. *The first Chern number $c_1(A_-)$ of A_- is $k+1$ and the first Chern number $c_1(A_+)$ of A_+ is $n-k+1$.*

Before proving this lemma, we investigate all weights at a vertex in the moment polytope and weights at a new vertex after applying the cut. By the geometry of the Grassmannians, all vertices are in either a triangular face or a square face.

Configuration 1:



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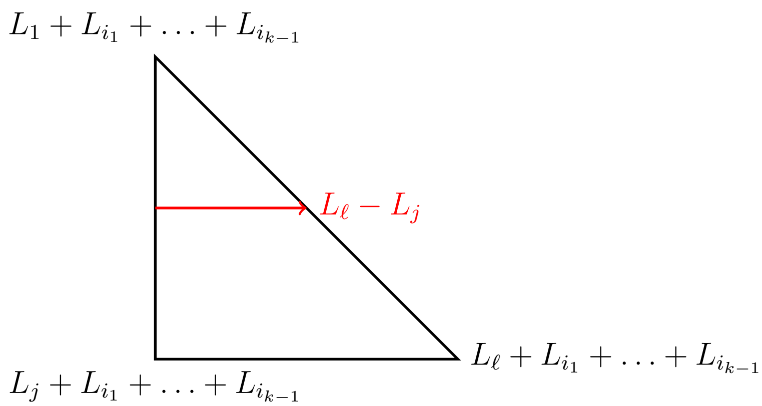
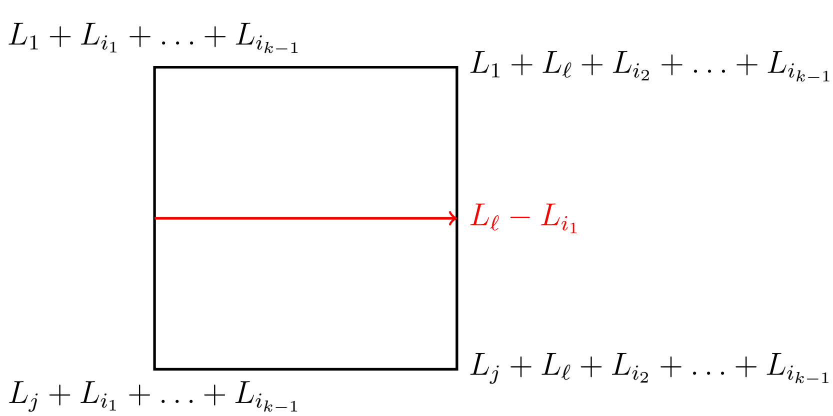


Configuration 3:



Type	Weights	Condition	Number of edges
A	$\pm(L_j - L_1)$	$+$ sign for M_-, $-$ sign for M_+	1
B	$L_j - L_{i_s}$	$s = 1, \ldots, k-1$	$k-1$
C	$L_\ell - L_j$	$\ell \neq 1, i_1, \ldots, i_{k-1}, j$	$n-k-1$
D	$L_\ell - L_{i_s}$	$\ell \neq 1, i_1, \ldots, i_{k-1}, j$ and $s = 1, \ldots, k-1$	$(n-k-1)(k-1)$

Table 4.1: Weights at v

Proof. (of Lemma 4.5.5) Let $v_1 = L_1 + L_{i_1} + \ldots + L_{i_{k-1}}$ and $v_2 = L_j + L_{i_1} + \ldots + L_{i_{k-1}}$, for some $j \neq i_s$, be the two vertices on the moment polytope of M, connecting the edge whose preimage is a holomorphic line representing the class A. Let v be the new vertex on the edge between v_1 and v_2 after applying the smooth cut. We list the weights at v in Table 4.1; they are the same in both M_- and M_+, unless otherwise noted. The edges of type A are edges coming out of the slice, the edges of type B are from Configuration 1, the edges of type C are from Configuration 2, and the edges of type D are from Configuration 3. The total number of edges is

$$1 + k - 1 + n - k - 1 + (n-k-1)(k-1) = n - 1 + nk - n - k^2 + 1 = k(n-k)$$

which is the correct degree of vertex v.

For a fixed point $p \in M^{\mathbb{T}}$, the first Chern class $c_1^{\mathbb{T}}|_p$ is equal to the sum of the $\mathbb{T}$-weights at that fixed point. The $\mathbb{T}$-weights precisely correspond to the edge directions at the corresponding vertex in the moment polytope [13, Equation (C.13)].

Case of M_-: We consider corresponding edges of v and v_2 in the moment polytope of M_-. By the localization formula due to Atiyah-Bott and Berline-Vergne

weights at v	weights at v_2	Note	Contri.	Multiplicity
$L_j - L_1$	$L_1 - L_j$	weights out of slice	2	1
$L_j - L_{i_s}$	$L_1 - L_{i_s}$	$s = 1, \ldots, k-1$	1	$k-1$
$L_\ell - L_j$	$L_\ell - L_j$	$\ell \neq 1, i_1, \ldots, i_{k-1}, j$	0	$n-k-1$
$L_\ell - L_{i_s}$	$L_\ell - L_{i_s}$	$\ell \neq 1, i_1, \ldots, i_{k-1}, j$ $s = 1, \ldots, k-1$	0	$(n-k-1)(k-1)$

Table 4.2: Weight contribution of v and v_2 to $c_1(A_-)$ in M_-

weights at v	weights at v_1	Note	Contri.	Multiplicity
$L_1 - L_j$	$L_j - L_1$	weights out of slice	2	1
$L_j - L_{i_s}$	$L_j - L_{i_s}$	$s = 1, \ldots, k-1$	0	$k-1$
$L_\ell - L_j$	$L_\ell - L_1$	$\ell \neq 1, i_1, \ldots, i_{k-1}, j$	1	$n-k-1$
$L_\ell - L_{i_s}$	$L_\ell - L_{i_s}$	$\ell \neq 1, i_1, \ldots, i_{k-1}, j$ $s = 1, \ldots, k-1$	0	$(n-k-1)(k-1)$

Table 4.3: Weight contribution of v and v_1 to $c_1(A_+)$ in M_+

([2],[3]), we can compute $c_1(A_-)$ as follows:

$$c_1(A_-) = \int_{S^2} c_1^{\mathbb{T}} = \frac{\text{weights at } v}{L_j - L_1} + \frac{\text{weights at } v_2}{L_1 - L_j}.$$

We list the edges of v and v_2 in Table 4.2 below along with its contribution to $c_1(A_-)$.

By Table 4.2, $c_1(A_-) = 2 + k - 1 = k + 1$.

Case of M_+: We consider corresponding edges of v and v_1 in the moment polytope of M_+. Due to ABBV localization ([2],[3]), we have

$$c_1(A_+) = \int_{S^2} c_1^{\mathbb{T}} = \frac{\text{weights at } v}{L_1 - L_j} + \frac{\text{weights at } v_1}{L_j - L_1}.$$

We list the edges of v and v_1 in Table 4.3 below along with its contribution to $c_1(A_+)$. By Table 4.3, $c_1(A_+) = 2 + n - k - 1 = n - k + 1$.

□

Lemma 4.5.6. *The standard almost complex structure J_0 is regular for class $A_- \in H_2(M_-;\mathbb{Z})$ and $A_+ \in H_2(M_+;\mathbb{Z})$*

Proof. To show that J_0 is regular for the class A_-, it suffices to pick a representative u of A_- such that D_u is onto. We pick the representative $u : \mathbb{CP}^1 \to M_-$ of A_- that is the simple curve whose moment map image is the edge connecting v and v_2 as in the previous lemma. In this case, the splitting of $TM_-|_{u(\mathbb{CP}^1)}$ into line bundles is equivariant. Each pair of corresponding edges between v and v_2 form a plane and the restriction of this plane to the curve is an equivariant line bundle [15, page 288]. From the calculations of the first Chern number $c_1(A_-)$, every equivariant line bundle in $u^*(TM_-)$ has Chern number at least 0. Therefore by Lemma 2.5.2 u is regular and it follows that J_0 is regular for A_-. Similarly, J_0 is regular for A_+ using v and v_1 in the previous lemma. $\square$

Next we compute the first Chern number of spheres corresponding to new edges on the slice in the moment polytope of the symplectic cut. The following lemma tells us that the first Chern number of new edges on the slice is nonnegative.

Lemma 4.5.7. *Let B be the homology corresponding to the sphere of an edge in the slice of the moment polytope. Then $c_1(B) \geq 0$ for both M_- and M_+.*

Proof. Let $E = (v_1, v_2)$ be an edge in the slice of the moment polytope after the cut and let B be its second homology. Again by the geometry of the Grassmanniann, new edges are formed from either Configuration 1, 2, or 3 above. We use labeled edges in those configurations.

Configuration 1: Suppose $E = (v_1, v_2)$ is the new edge with left vertex v_1 and

weights at v_1	weights at v_2	Note	Contri.	Multiplicity
$L_j - L_{i_1}$	$L_{i_1} - L_j$	sphere connecting them	2	1
$\pm(L_j - L_1)$	$\pm(L_{i_1} - L_1)$	weights out of slice	1 or -1	1
$L_j - L_{i_s}$	$L_{i_1} - L_{i_s}$	$s = 2, \dots, k-1$	1	$k-2$
$L_\ell - L_j$	$L_\ell - L_{i_1}$	$\ell \neq 1, i_1, \dots, i_{k-1}, j$	-1	$n-k-1$
$L_\ell - L_{i_1}$	$L_\ell - L_j$	$\ell \neq 1, i_1, \dots, i_{k-1}, j$ $s = 2, \dots, k-1$	1	$n-k-1$
$L_\ell - L_{i_s}$	$L_\ell - L_{i_s}$	$\ell \neq 1, i_1, \dots, i_{k-1}, j$ $s = 2, \dots, k-1$	0	$(n-k-1)(k-2)$

Table 4.4: Weight contribution of v_1 and v_2 to $c_1(B)$ in M_- and M_+ in Configuration 1

weights at v_1	weights at v_2	Note	Contri.	Multiplicity
$L_\ell - L_j$	$L_j - L_\ell$	sphere connecting them	2	1
$\pm(L_j - L_1)$	$\pm(L_\ell - L_1)$	weights out of slice	1 or -1	1
$L_j - L_{i_s}$	$L_\ell - L_{i_s}$	$s = 1, \dots, k-1$	-1	$k-1$
$L_r - L_j$	$L_r - L_\ell$	$r \neq 1, i_1, \dots, i_{k-1}, j, \ell$	1	$n-k-2$
$L_\ell - L_{i_s}$	$L_j - L_{i_s}$	$s = 1, \dots, k-1$	1	$k-1$
$L_r - L_{i_s}$	$L_r - L_{i_s}$	$r \neq 1, i_1, \dots, i_{k-1}, j, \ell$ $s = 1, \dots, k-1$	0	$(n-k-2)(k-1)$

Table 4.5: Weight contribution of v_1 and v_2 to $c_1(B)$ in M_- and M_+ in Configuration 2

right vertex v_2. We list the edges from v_1 and v_2 along with its contribution to $c_1(B)$ in Table 4.4, similar to those in the proof of Lemma 4.5.5.

Thus, $c_1(B) = k \pm 1 \geq 0$.

Configuration 2: Suppose $E = (v_1, v_2)$ is the new edge with left vertex v_1 and right vertex v_2. We again list the edges from v_1 and v_2 along with its contribution to $c_1(B)$ in Table 4.5.

Thus, $c_1(B) = n - k \pm 1 \geq 0$.

weights at v_1	weights at v_2	Note	Contri.	Multiplicity
$L_\ell - L_{i_1}$	$L_{i_1} - L_\ell$	sphere connecting them	2	1
$\pm(L_j - L_1)$	$\pm(L_j - L_1)$	weights out of slice	0	1
$L_j - L_{i_s}$ $L_j - L_{i_1}$	$L_j - L_{i_s}$ $L_j - L_\ell$	$s = 2, \dots, k-1$ $i_s = 1$ or ℓ	0 1	$k-2$ 1
$L_r - L_j$ $L_\ell - L_j$	$L_r - L_j$ $L_{i_1} - L_j$	$r \neq 1, i_1, \dots, i_{k-1}, j, \ell$ extra comb.	0 1	$n-k-2$ 1
$L_r - L_{i_s}$	$L_r - L_{i_s}$	$s = 2, \dots, k-1$ $r \neq 1, i_1, \dots, i_{k-1}, j, \ell$	0	$(n-k-2)(k-2)$
$L_r - L_{i_1}$	$L_r - L_\ell$	$r \neq 1, i_1, \dots, i_{k-1}, j, \ell$	1	$n-k-2$
$L_\ell - L_{i_s}$	$L_{i_1} - L_{i_s}$	$s = 2, \dots, k-1$	1	$k-2$

Table 4.6: Weight contribution of v_1 and v_2 to $c_1(B)$ in M_- and M_+ in Configuration 3

Configuration 3: Suppose $E = (v_1, v_2)$ is the new edge with left vertex v_1 and right vertex v_2. We again list the edges from v_1 and v_2 along with its contribution to $c_1(B)$ in Table 4.6. Thus, $c_1(B) = n \geq 0$. $\square$

A symplectic manifold is **semi-positive** if $c_1(A) \geq 0$ for all spherical homology class $A \in H_2(M; \mathbb{Z})$ such that $\omega(A) > 0$ and $c_1(A) \geq 3 - n$. An almost complex structure J is called **semi-positive** if every J-holomorphic curve has nonnegative first Chern number. This notion of semi-positivity is equivalent to the notion of **nef** for algebraic curves in algebraic geometry. The semi-positive property of J is a generic property. The regularity property of J is also a generic property. Thus, we can choose the standard almost complex structure J_0 and only need to show the semi-positivity property for this J_0. The semi-positivity property allows us to compactify the moduli space $\mathcal{M}^*_{A_-}(M_-, J_0)$ and $\mathcal{M}^*_{A_-}(M_-, J_0)$. Compactness of the moduli space guarantees that Gromov-Witten invariants are well-defined [33, chapter 5].

In general Hamiltonian $\mathbb{T}$-spaces enjoy the topological property of **equivari-**

ant formality over $\mathbb{Q}$. When the fixed points are isolated, equivariant formality also holds over $\mathbb{Z}$. So in the case of complex Grassmannians and their symplectic cuts, we have a surjection

$$H^*_{\mathbb{T}}(M;\mathbb{Z}) \to H^*(M;\mathbb{Z}).$$

Dually, in homology, we know that $H_2(M;\mathbb{Z})$ is spanned by T-invariant submanifolds of M. For Grassmannians and their cuts, then, $H_2(M;\mathbb{Z})$ is spanned by the homology classes of spheres corresponding to edges in the moment polytope. Thus, if we show that for the first Chern number of the homology classes of spheres corresponding to edges in the moment polytope is nonnegative, we can guarantee that symplectic cuts of complex Grassmannians are semi-positive.

Lemma 4.5.8. *The symplectic cuts M_- and M_+ are semi-positive for J_0.*

Proof. It suffices to show that the first Chern number of the homology classes of spheres corresponding to edges in the moment polytope is nonnegative. There are three kinds of edges in M_- and M_+:

(i) edges whose moment map preimages are representatives of homology class A. It is well known that $c_1(A) = n$, which is nonnegative.

(ii) edges whose moment map preimages are representatives of homology class A_- and A_+. By Lemma 4.5.5, $c_1(A_-) = k+1$ and $c_1(A_+) = n-k+1$, which are also nonnegative.

(iii) edges with homology B in the slice. By Lemma 4.5.7, $c_1(B) \geq 0$.

Therefore, M_- and M_+ are semi-positive for J_0. $\square$

Let us consider the evaluation map

$$\mathrm{ev}^2_{J_0} : \overline{\mathcal{M}_{A_-,2}(M_-, J_0)} \to M_-^2.$$

For a generic point $p \in M_-$, we want to find a compact complex submanifold $X \subset M_-$ that satisfies the following conditions:

(1) The dimensionality condition $\dim_{\mathbb{R}} \mathcal{M}_{A_-,2}(M_-, J_0) + \dim_{\mathbb{R}} X = 2\dim_{\mathbb{R}} M_-$ is satisfied,

(2) The number of holomorphic curves in $\mathcal{M}_{A_-}(M_-, J_0)/PSL(2, \mathbb{C})$ that pass through p and X is different from zero,

(3) The evaluation map $\mathrm{ev}^2_{J_0}$ is transverse to $(\{p\} \times X) \subset M_-^2$.

This will guarantee that the Gromov-Witten invariant $\mathrm{GW}^{J_0}_{A_-,2}(\mathrm{PD}[p], \mathrm{PD}[X]))$ is different from zero.

Likewise for M_+, we consider the evaluation map

$$\mathrm{ev}^2_{J_0} : \overline{\mathcal{M}_{A_+,2}(M_+, J_0)} \to M_+^2.$$

For a generic point $p \in M_+$, we want to find a compact complex submanifold $Y \subset M_+$ that satisfies the following conditions:

(1) The dimensionality condition $\dim_{\mathbb{R}} \mathcal{M}_{A_+,2}(M_+, J_0) + \dim_{\mathbb{R}} Y = 2\dim_{\mathbb{R}} M_+$ is satisfied,

(2) The number of holomorphic curves in $\mathcal{M}_{A_+}(M_+, J_0)/PSL(2, \mathbb{C})$ that pass through p and Y is different from zero,

(3) The evaluation map $\mathrm{ev}^2_{J_0}$ is transverse to $(\{p\} \times Y) \subset M^2_+$.

This will guarantee that the Gromov-Witten invariant $\mathrm{GW}^{J_0}_{A_+,2}(\mathrm{PD}[p], \mathrm{PD}[Y]))$ is different from zero.

We claim that the submanifolds

$$X = \left\{ V \in \mathcal{G}r(k,n) \,\middle|\, V \subset \mathbb{C}[e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n] \right\}$$

and

$$Y = \left\{ V \in \mathcal{G}r(k,n) \,\middle|\, \mathbb{C}[e_1] \subset V \right\}$$

satisfy all these conditions. Note that the complex submanifold X is isomorphic to the Grassmannian submanifold $\mathcal{G}r(k, n-1)$ and Y is isomorphic to $\mathcal{G}r(k-1, n-1)$.

Lemma 4.5.9. *If J is integrable, then each J-holomorphic curve is counted in $\mathrm{GW}^{J_0}_{A_-,2}(\mathrm{PD}[p], \mathrm{PD}[X]))$ and $\mathrm{GW}^{J_0}_{A_+,2}(\mathrm{PD}[p], \mathrm{PD}[Y]))$ with a positive sign. In particular,*

$$\mathrm{GW}^{J_0}_{A_-,2}(\mathrm{PD}[p], \mathrm{PD}[X])) \geq 0$$

and

$$\mathrm{GW}^{J_0}_{A_+,2}(\mathrm{PD}[p], \mathrm{PD}[Y])) \geq 0$$

Proof. If J_0 is integrable, then the evaluation map

$$\mathrm{ev}^2_{J_0} : \overline{\mathcal{M}_{A_-,2}(M_-, J)} \to M^2_-$$

is holomorphic and the sign is simply the product of signs which are attached to each component via the evaluation. Therefore being holomorphic preserves orientation everywhere. Hence, all the curves count with $+1$. A similar result holds for M_+. $\square$

We now check item (1) from our list.

Lemma 4.5.10. *The manifold* $X = \{V \in \mathcal{G}r(k,n) | V \subset \mathbb{C}[e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n]\}$ *satisfies the dimensionality condition*

$$\dim_{\mathbb{R}} \mathcal{M}_{A_-,2}(M_-, J_0) + \dim_{\mathbb{R}} X = 2 \dim_{\mathbb{R}} M_-.$$

Likewise, the manifold $Y = \{V \in \mathcal{G}r(k,n) | \mathbb{C}[e_1] \subset V\}$ *satisfies the dimensionality condition*

$$\dim_{\mathbb{R}} \mathcal{M}_{A_+,2}(M_+, J_0) + \dim_{\mathbb{R}} Y = 2 \dim_{\mathbb{R}} M_+.$$

Proof. We have $\dim_{\mathbb{R}} X = 2k(n-k-1)$ and

$$\begin{aligned}
\dim_{\mathbb{R}} X + \dim_{\mathbb{R}} M_- + 2c_1(A_-) + 2 \cdot 2 - 6 \quad &= 2k(n-k-1) + 2k(n-k) \\
&+2(k+1) - 2 \\
&= 2k(n-k-1) + 2k(n-k) + 2k \\
&= 2k(n-k) + 2k(n-k) \\
&= 2(2k(n-k)) \\
&= 2 \dim_{\mathbb{R}} M_-
\end{aligned}$$

Likewise, we have $\dim_{\mathbb{R}} Y = 2(k-1)(n-k)$ and

$$\begin{aligned}
\dim_{\mathbb{R}} Y + \dim_{\mathbb{R}} M_+ + 2c_1(A_+) + 2 \cdot 2 - 6 \quad &= 2(k-1)(n-k) + 2k(n-k) \\
&+2(n-k+1) - 2 \\
&= 2(k-1)(n-k) + 2k(n-k) \\
&+2(n-k) \\
&= 2k(n-k) + 2k(n-k) \\
&= 2(2k(n-k)) \\
&= 2 \dim_{\mathbb{R}} M_+
\end{aligned}$$

□

Lemma 4.5.11. *Let V be a complex k-plane spanned by the rows of matrix A*

$$A = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & \dots & \vdots \\ v_{k1} & \dots & v_{kn} \end{pmatrix}$$

such that $\left(\frac{1}{n-k} \sum_{i=1}^{k} |v_{i1}|^2 \right) - \frac{1}{k} = \alpha$. *Then V can be spanned by*

$$B = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ 0 & w_{22} & \dots & w_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & w_{k2} & \dots & w_{kn} \end{pmatrix}$$

such that $\frac{1}{n-k} |w_{11}|^2 - \frac{1}{k} = \alpha$.

Note that the rows of a matrix representation of a complex k-plane are orthonormal.

Proof. We will transform matrix A in the following steps:

- *Step 1*: Without loss of generality, suppose $v_{11} \neq 0$. Apply row reduction to matrix A to obtain

$$A' = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ 0 & v'_{22} & \dots & v'_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & v'_{k2} & \dots & v'_{kn} \end{pmatrix}$$

- *Step 2*: Apply Gram-Schmidt algorithm from the bottom row to the top row of A' to obtain:

$$B = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ 0 & w_{22} & \dots & w_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & w_{k2} & \dots & w_{kn} \end{pmatrix}$$

- *Step 3*: Then $\frac{1}{n-k}|w_{11}|^2 - \frac{1}{k} = \alpha$ because under the identification between complex k-planes and hermitian matrices

$$\left(\frac{1}{n-k}\sum_{i=1}^{k}|v_{i1}|^2\right) - \frac{1}{k} = f(A) = f(B) = \frac{1}{n-k}|w_{11}|^2 - \frac{1}{k} = \alpha$$

where f is the moment map corresponding to circle weight L_1.

$\square$

Next we establish the existence of an appropriate J_0-holomorphic curve.

Proposition 4.5.12. *For a generic point $p \in M_-$ there exists a J_0-holomorphic curve in $\mathcal{M}_{A_-}(M_-, J_0)/PSL(2,\mathbb{C})$ through p and X. Similarly, for a generic point $p \in M_+$ there exists a J_0-holomorphic curve in $\mathcal{M}_{A_+}(M_+, J_0)/PSL(2,\mathbb{C})$ through p and Y.*

Proof. Denote the open dense set $O = f^{-1}\left(-\frac{1}{k}, \frac{1}{n-k} - \epsilon\right)$ and let $p \in O$. Then there exists $\alpha \in \left(-\frac{1}{k}, \frac{1}{n-k} - \epsilon\right)$ such that $p \in f^{-1}(\alpha)$ and we can write

$$p = \operatorname{span}(v_1, \dots, v_k)$$

where $\sum_{i=1}^{n}|v_{ji}|^2 = 1$, $\sum_{i=1}^{n} v_{ji}\overline{v_{\ell i}} = 0$ and $\left(\frac{1}{n-k}\sum_{i=1}^{k}|v_{i1}|^2\right) - \frac{1}{k} = \alpha$. By Lemma

4.5.11, p is spanned by the rows of

$$B = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ 0 & w_{22} & \dots & w_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & w_{k2} & \dots & w_{kn} \end{pmatrix}$$

with $\dfrac{1}{n-k}|w_{11}|^2 - \dfrac{1}{k} = \alpha$. Let

$$S = \Big\{ V \in \mathcal{G}r(k,n) \mid \operatorname{span}(e_2, e_3, \dots, e_k) \subset V \subset \operatorname{span}(e_1, e_2, \dots, e_k, e_{k+1}) \Big\} \subset \mathcal{G}r(k,n)$$

and S is the image of a J-holomorphic curve in the class of A in $\mathcal{G}r(k,n)$. Let q be a complex k-plane in S spanned by $(e_2, \dots, e_k, w_{11}e_1 + ce_{k+1})$ for some $c \in \mathbb{C}$ satisfying $|w_{11}|^2 + |c|^2 = 1$. Let $\tilde{S}$ be the resulted J_0-holomorphic curve obtained from S in the class of A_- after applying the symplectic cut.

Let

$$H = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & SU(n-1) \end{pmatrix}$$

be the subgroup of $SU(n)$ that fixes the first coordinate. Then by the previous construction of B and transitivity action of $SU(n-1)$ on the remaining $n-1$ coordinates, there exists $g \in H$ such that $g \cdot q = p$. Thus $g \cdot \tilde{S}$ is the image of a J-holomorphic curve in the class of A_- that contains p.

Since H is a Lie subgroup, there exists a continous 1-parameter family of H that takes the identity element $e \in H$ to $g \in H$. This induces a smooth 1-parameter family of holomorphic diffeomorphisms from the identity map ϕ_e to ϕ_g. This implies that ϕ_g is homotopic to the identity map. Since homology is invariant under homotopic maps, we have that

$$[\phi_g(\tilde{S})] = [\phi_e(\tilde{S})] = [\tilde{S}] = A_-.$$

Note that $f^{-1}\left(-\frac{1}{k}\right) = X$. Since the action of H preserves the first component, $g \cdot S$ goes through both the point p and the submanifold X. The proof of the second statement for M_+ is analogous. $\square$

Definition 4.5.13. A d-dimensional **pseudocycle** in a manifold Y is a smooth map

$$f : V \to Y$$

defined on an oriented d-dimensional manifold V such that its image $f(V)$ has compact closure and its limit set Ω_f has dimension $\leq d-2$. Here

$$\Omega_f = \bigcap_{\substack{K \subset V \\ K \text{ compact}}} \overline{f(V \setminus K)}$$

is the set of all limit points of sequences $f(y_\nu)$ where y_ν has no convergent subsequence in V.

Observation 4.5.14. *The identity map $f : \mathcal{G}r(k, n-1) \to M_-$ from $\mathcal{G}r(k, n-1)$ to X is a pseudocycle dual to the cohomology class $\mathrm{PD}[X]$. Similarly, the identity map $g : \mathcal{G}r(k-1, n-1) \to M_+$ from $\mathcal{G}r(k-1, n-1)$ to Y is a pseudocycle dual to the cohomology class $\mathrm{PD}[Y]$.*

This verifies the required transversality. We now put conditions (1), (2), and (3) together to verify the existence of a non-zero Gromov-Witten invariant.

Lemma 4.5.15. $\mathrm{GW}^{J_0}_{A_-,2}(\mathrm{PD}[p], \mathrm{PD}[X])) \neq 0$ *and* $\mathrm{GW}^{J_0}_{A_+,2}(\mathrm{PD}[p], \mathrm{PD}[X])) \neq 0$.

Proof. By [33, Exercise 7.1.2], $\mathrm{GW}^{J_0}_{A_-,2}(\mathrm{PD}[p], \mathrm{PD}[X]))$ is the intersection number of $\mathrm{ev}^2_{J_0}$ with a product cycle $f_1 \times f_2$ where f_1 is the identity map of a generic point

p and f_2 is the map defined in Observation 4.5.14 above. By Proposition 4.5.12, the intersection of $\mathrm{ev}^2_{J_0}$ with the the product cycle $f_1 \times f_2$ is at least 1. Hence,

$$\mathrm{GW}^{J_0}_{A_-,2}(\mathrm{PD}[p], \mathrm{PD}[X])) \geq 1.$$

The proof for $\mathrm{GW}^{J_0}_{A_+,2}(\mathrm{PD}[p], \mathrm{PD}[X])) \geq 1$ is analogous. □

We have proved that for either symplectic cuts of Grassmannian manifolds there is a non-vanishing Gromov-Witten invariant with one of its constraints Poincaré dual to the class of a point. By Theorem 2.4.2, the Gromov width of such manifolds is bounded above by the symplectic area $\omega(A_-)$ and $\omega(A_+)$ of the homology class A_- and A_+ respectively. Proposition 4.5.3 establishes the lower bounds for M_- and M_+ that agree with these upper bounds. In summary we have established the main result for this section:

Theorem 4.5.16. *The Gromov width of M_- is*

$$\mathrm{Gwidth}(M_-) = \omega(A_-),$$

and the Gromov width of M_+ is

$$\mathrm{Gwidth}(M_+) = \omega(A_+).$$

Thus the monotonicity condition is satisfied:

$$\mathrm{Gwidth}(\widetilde{M}) \leq \mathrm{Gwidth}(M)$$

for all symplectic cuts $\widetilde{M}$ of M by a circle weight L_i.

For our special cuts by circle weight L_i, we have a strong relationship,

$$\mathrm{Gwidth}(M) = \mathrm{Gwidth}(M_-) + \mathrm{Gwidth}(M_+),$$

where M_- and M_+ are resulting cut spaces of M. This additivity does not hold for general smooth cuts of $\mathcal{G}r(k, n)$.

Proposition 4.5.17. *The Gromov width is not additive.*

Proof. In the case of an equivariant blow-up at a fixed point as in Figure 4.2 the Gromov width is not additive. The sum of the Gromov widths of M_- and M_+ is strictly greater than the Gromov width of M because the Gromov width of the blow-up manifold M_- is the same as the Gromov width of M. $\square$

We now wish to show that the Gromov-Witten invariant is exactly 1. This way there is a natural inheritance of J-holomorphic curves in the class of A from upstairs to J-holomorphic curves in the class A_- or A_+.

Proposition 4.5.18. *For $\widetilde{M} = M_-$ or M_+, $\widetilde{A} = A_-$ or A_+, $Z = X$ or Y, we have*

$$\mathrm{GW}^{J_0}_{\widetilde{A},2}(\mathrm{PD}[p], \mathrm{PD}[Z])) = 1.$$

Proof. Suppose that $\mathrm{GW}^{J_0}_{\widetilde{A},2}(\mathrm{PD}[p], \mathrm{PD}[Z])) > 1$. Fix two marked points, z_1 and z_2 on S^2 and let u be the J-holomorphic curve in the class $\widetilde{A}$ defined as in the proof of Proposition 4.5.12. Let P and Q be the images of z_1 and z_2 under u in $\widetilde{M}$ so that $u(z_1) \notin Z$ and $u(z_2) \in Z$, denote C the image of u. Let u' be another distinct J-holomorphic curve in that class $\widetilde{A}$ whose image contains P and Q, and denote C' the image of u'.

Let

$$S = \left\{ V \in \mathcal{G}r(k,n) \mid \mathrm{span}(e_2, e_3, \ldots, e_k) \subset V \subset \mathrm{span}(e_1, e_2, \ldots, e_k, e_{k+1}) \right\} \subset \mathcal{G}r(k,n)$$

and S is the image of a J-holomorphic curve in the class of A in $\mathcal{G}r(k,n)$ and let S' be the J-holomorphic curve obtained after the symplectic cut.

Let $g \in SU(n-1)$ such that $g \cdot C = S'$. By the local normal form, there exists an equivariant ball B centered around $g \cdot Q$ of capacity $\omega(\widetilde{A})$ containing the curve

S'. B is the image of a J-holomorphic embedding ϕ of a ball $B^{2n}(r)$ of radius r into $\widetilde{M}$. Since $g \cdot C'$ is a J-holomorphic curve of capacity $\omega(\widetilde{A})$ containing the center $g \cdot Q$, $g \cdot C'$ ia also contained inside B. Let $D = B^{2n}(r) \cap \phi^{-1}(g \cdot C')$ and we claim that $\phi(D) = g \cdot C'$. On $B^{2n}(r)$, the almost complex structure is standard. Therefore the preimage D is holomorphic, hence minimal surface in $B^{2n}(r)$ with boundary in $\delta B^{2n}(r)$. By the monotonicity formula, the area of D is at least $\pi r^2 = \omega(\widetilde{A})$. This implies area of $v(D)$ equals $\omega(g \cdot C')$. Hence, $\phi(D) = g \cdot C'$. Thereofore, $g \cdot C'$ contains inside B.

Since there is a unique J-holomorphic curve going through two points inside B, namely $g \cdot P$ and $g \cdot Q$, $g \cdot C$ and $g \cdot C'$ are the same curve. This implies that $C = C'$, which is a contradiction. $\square$

BIBLIOGRAPHY

[1] M. F. Atiyah. Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* 14 (1982), 1-15.

[2] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[3] N. Berline and M. Vergne. Classes caractéristiques éequivariantes. formules de localisation en cohomologie équivariante. *C.R. Acad. Sci. Paris Sér. I Math.*

[4] A. S. Buch, A. Kresch and H. Tamvakis. Gromov-Witten invariants on Grassmanians. *J. Amer. Math. Soc.* **16** (2003), no. 4, 901-915 (electronic), available at http://lanl.arxiv.org/abs/math/0306388.MR1992829 (2004h:14060).

[5] A. Cannas da Silva. Symplectic Toric Manifolds, in M. Audin, A. Cannas da Silva, and E. Lerman, Symplectic geometry of integrable Hamiltonian systems. *Advanced courses in mathematics*, CRM Barcelona, Birkhäuser Verlag, Basel, 2003.

[6] A. Cannas da Silva. Lectures on Symplectic Geometry. *Springer*. Berlin, 2001.

[7] A. Caviedes–Castro. Upper bound for the Gromov width of coadjoint orbits of compact Lie groups. University of Toronto. Ph.D. thesis, 2014.

[8] D. Cristofaro-Gardiner. Symplectic embeddings from concave toric domains into convex ones. Arxiv-1409.4378.

[9] T. Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France* 116 (1988), 315-339.

[10] W. Fulton and J. Harris. Representation theory. A first course. *Graduate Texts in Mathematics, Reading in Mathematics*. 129. New York-Verlag.

[11] M. Goresky, R. Kottwitz and R. MacPherson. Equivariant cohomlogy, Koszul duality, and the localization theorem. *Inventiones Mathematicae*, 131:25-83, 1997.

[12] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* 82 (1985), no. 2, 307-347.

[13] V. Guillemin, V. Ginburg and Y. Karshon. Moment maps, cobordisms, and Hamiltonian group actions. *Mathematical Surveys and Monographs*, 98, American Mathematical Society, Providence, RI, 2002.

[14] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping, *Invent. Math.* 67 (1982), 491-513, Contemp. Math. , 460,

[15] V. Guillemin and C. Zara. 1-Skeleta, Betti numbers, and equivariant cohomology. *Duke Math Journal*. Vol. 107, Number 2 (2001), 283-349.

[16] T. S. Holm. Act globally, compute locally: group actions, fixed points, and localization, *Toric topology*, 179-195. Amer. Math. Soc., Providence, RI, 2008.

[17] J. Hu, Gromov-Witten invariants of blow-ups along points and curves. *Math. Z.*, 233 (2000), no. 4, 709739.

[18] M. Hutchings. Qualitative embedded contact homology. *Journal of Differential Geometry*, Vol. 88, 231-266, 2011.

[19] Y. Karshon and L. Kessler. Distinguishing symplectic blowups of the complex projective plane. Submitted.

[20] Y. Karshon and L. Kessler. Circle and Torus Actions on Equal Symplectic Blow-ups of $\mathbb{CP}^2$. *Math. Res. Letter*. **14**, 807-823, 2007.

[21] Y. Karshon, L. Kessler and M. Pinsonnault. Symplectic blowups of the complex projective plane and counting toric actions. A preliminary version is posted on the authors' websites.

[22] Y. Karshon and S. Tolman. The Gromov width of complex Grassmannians. *Algebr. Geom. Topol.*, 5:911922 (electronic), 2005.

[23] F. Kirwan. Cohomology of quotients in symplectic and algebraic geometry, volume 31 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1984.

[24] J. Li and G. Tian. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), 47-83, 1998.

[25] E. Lerman. Symplectic cuts. *Math. Research Lett.* **2** (1995), 247-258.

[26] G. Lu. Gromov-Witten invariants and pseudo symplectic capacities. *Israel J. Math.* 156 (2006), 1–63.

[27] G. Lu. Symplectic capacities of toric manifolds and related results. *Nagoya Math. J.* 181 (2006), 149-184.

[28] M. Hutchings. Recent progress on symplectic embedding problems in four dimensions. arXiv:1101.1069v2.

[29] A. Mandini and M. Pabiniak. On the Gromov width of polygon spaces. *Transformation Groups*. Springer US. 2017.

[30] D. McDuff. From symplectic deformation to isotopy. *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), 85-99. First int. Press Lect. Ser., I, internat. Press, Cambridge, MA, 1998.

[31] D. McDuff. Symplectic embeddings of 4-dimensional ellipsoids. *J Topology* 2:1-22.

[32] D. McDuff and D. Salamon. J-holomorphic curves and symplectic topology. *American Mathematical Society Colloquium Publications*, vol. 52, American Mathematical Society, Providence, RI, 2004.

[33] D. McDuff and D. Salamon. J-holomorphic curves and quantum cohomology. *Univ. Lect. Ser. 6*, Amer. Math. Soc. Providence, RI (1994).

[34] D. McDuff and L. Polterovich. Symplectic packings and algebraic geometry. *Invent. Math.*, **115** (1994), 405-425.

[35] T. Oda. Convex bodies and algebraic geometry. *An introduction to the Theory of Toric Varieties*. Ergeb. Math. Grenzgeb. (2). 15. Springer-Verlag, Berlin, 1988.

[36] M. Pabiniak. Hamiltonian torus action in equivariant cohomology and symplectic topology. Cornell Ph.D. Dissertation 2012.

[37] Y. Ruan. Virtual neighborhoods and pseudo-holomorphic curves. *Preceedings of 6th Gökova Geometry-Topology Conference*, 161-231, 1999.

[38] S. Tolman and J. Weitsman. On the cohomology rings of Hamiltonian T-spaces. In *Northern California Symplectic Geometry Seminar*, volume 196 of

Amer. Math. Soc. Trans. Ser. 2, pages 251-258. Amer. Math. Soc., Providence, RI, 1999.

[39] L. Traynor. Symplectic Packing Constructions. J. Diff. Geom. 42 No. 2 (1995) 411429.

[40] A. Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Adv. Math.* 6 (1971) 329-346.

[41] M. Zoghi. The Gromov width of coadjoint orbits of compact Lie Groups (2010), 91 pp. Thesis (Ph.D.)–University of Toronto (Canada).